

A maximal regularity estimate for the non-stationary Stokes equation in the strip

Antoine Choffrut, Camilla Nobili and Felix Otto

March 28, 2017

Abstract

In a d -dimensional strip with $d \geq 2$, we study the non-stationary Stokes equation with no-slip boundary condition in the lower and upper plates and periodic boundary condition in the horizontal directions. In this paper we establish a new maximal regularity estimate in the real interpolation norm

$$\|f\|_{(0,1)} = \inf_{f=f_0+f_1} \left\{ \left\langle \sup_{0 < z < 1} |f_0| \right\rangle + \left\langle \int_0^1 |f_1| \frac{dz}{(1-z)z} \right\rangle \right\},$$

where the brackets $\langle \cdot \rangle$ denotes the horizontal-space and time average. The norms involved in the definition of $\| \cdot \|_{(0,1)}$ are critical for two reasons: the exponents are borderline for the Calderón-Zygmund theory and the weight $1/z$ just fails to be Muckenhoupt. Therefore, the estimate is only true under horizontal bandedness condition, (i. e. a restriction to a packet of wave numbers in Fourier space). The motivation to express the maximal regularity in such a norm comes from an application to the Rayleigh-Bénard problem (see [1]).

Keywords. Non-stationary Stokes equations, no-slip boundary condition, maximal regularity, real interpolation .

Contents

1	Introduction	3
2	Maximal regularity in the strip	5
2.1	From the strip to the half space	5
2.2	Maximal regularity in the upper half space	6
2.3	Proof of Proposition 1	10
2.4	Proof of Proposition 2	10
2.5	Proof of Theorem 1	15
3	Proof of main technical lemmas	17
3.1	Proof of Lemma 1	17
3.2	Proof of Lemma 2	21
3.3	Proof of Lemma 3	22
4	Appendix	34
4.1	Preliminaries	34
4.2	Heat kernel: elementary estimates	38
5	Notations	39
	References	40

1 Introduction

In the d -dimensional strip $[0, L)^{d-1} \times [0, 1]$, $d \geq 2$, we consider the non-stationary Stokes equation for the vector field $u(x', z, t)$ and the scalar field $p(x', z, t)$

$$\begin{cases} \partial_t u - \Delta u + \nabla p = f & \text{for } 0 < z < 1, \\ \nabla \cdot u = 0 & \text{for } 0 < z < 1, \\ u = 0 & \text{for } z \in \{0, 1\}, \\ u = 0 & \text{for } t = 0, \end{cases} \quad (1)$$

where $x' \in [0, L)^{d-1}$ and $z \in [0, 1]$ indicate the spatial variables and $t \in \mathbb{R}^+$ denotes the time variable. In what follows it is important to distinguish the horizontal component $u' \in \mathbb{R}^{d-1}$ and the vertical component $u^z \in \mathbb{R}$ of the vector field u .

Motivated by an application to the Rayleigh-Bénard convection problem (see [1]), in this paper we establish the following maximal regularity estimate :

Theorem 1 (Maximal regularity in the strip).

There exists $R_0 \in (0, \infty)$ depending only on d and L such that the following holds. Let u, p, f satisfy the equation (1). Assume f is horizontally band-limited, i.e

$$\mathcal{F}' f(k', z, t) = 0 \text{ unless } 1 \leq R|k'| \leq 4 \text{ where } R < R_0. \quad (2)$$

Then,

$$\|(\partial_t - \partial_z^2)u'\|_{(0,1)} + \|\nabla' \nabla u'\|_{(0,1)} + \|\partial_t u^z\|_{(0,1)} + \|\nabla^2 u^z\|_{(0,1)} + \|\nabla p\|_{(0,1)} \lesssim \|f\|_{(0,1)}, \quad (3)$$

where $\|\cdot\|_{(0,1)}$ denotes the norm

$$\|f\|_{(0,1)} := \|f\|_{(R,(0,1))} = \inf_{f=f_0+f_1} \left\{ \left\langle \sup_{0 < z < 1} |f_0| \right\rangle + \left\langle \int_0^1 |f_1| \frac{dz}{(1-z)z} \right\rangle \right\}, \quad (4)$$

where f_0 and f_1 satisfy the bandedness assumption (2).

In the Theorem above, \mathcal{F}' denotes the horizontal Fourier transform, k' the conjugate variable of x' and the brackets $\langle \cdot \rangle$ stand for long-time and horizontal-space average. See Section 5 for notations.

The Theorem as stated above is used in this form in [1]. Alternatively, the theorem can be stated with the brackets $\langle \cdot \rangle$ denoting the integration

in $t > 0$, see Remark 2 at the beginning of Section 3. The maximal regularity in the strip is expressed in terms of the *interpolation* between the norms of $L^1 \left(dt dx' \frac{1}{z(1-z)} dz \right)$ and $L_z^\infty(L_{t,x'}^1)$, which are both borderline for the Calderón-Zygmund estimates. We notice that the norm of $L^1 \left(dt dx' \frac{1}{z(1-z)} dz \right)$ is critical both because of the exponent and the weight $\frac{1}{z(1-z)}$ are borderline, therefore, estimate (3) is only true under *bandedness assumptions* (i. e. a restriction to a packet of wave numbers in Fourier space). We observe that only bandedness in the *horizontal* variable x' is assumed and this is extremely convenient since the horizontal Fourier transform (or rather, series), with help of which bandedness is expressed, is compatible with the lateral periodic boundary conditions.

We notice that in the maximal regularity theory the no-slip boundary condition is a nuisance : As opposed to the no-stress boundary condition in the half space, the no-slip boundary condition does not allow for an extension by reflection to the whole space, and thereby the use of simple kernels or Fourier methods also in the normal variable. The difficulty coming from the the no-slip boundary condition in the non-stationary Stokes equations when deriving maximal regularity estimates is of course well-known; many techniques have been developed to derive Calderón-Zygmund estimates despite this difficulty. In the *half space* Solonnikov in [2] has constructed a solution formula for (1) with zero initial data via the Oseen and Green tensors. An easier and more compact representation of the solution to the problem (1) with zero forcing term and non-zero initial value was later given by Ukai in [3] by using a different method. Indeed he could write an explicit formula of the solution operator as a composition of Riesz' operators and solutions operator for the heat and Laplace's equation. This formula is an effective tool to get $L^p - L^q$ ($1 < q, p < \infty$) estimates for the solution and its derivatives. In the case of *exterior domains*, Maremonti and Solonnikov [4] derive $L^p - L^q$ ($1 < q, p < \infty$) estimates for (1), going through estimates for the extended solution in the half space and in the whole space. In particular in the half space they propose a decomposition of (1) with non-zero divergence equation. The book of Galdi [5] provides with a complete treatment of the classical theory and results on the non-stationary Stokes equations and Navier-Stokes equations.

In [1] the authors make substantial use of the estimate (3) in Theorem 1 to get bounds on the the Nusselt number, which is the natural measure of

the enhancement of upward heat flux for the Rayleigh-Bénard convection. There, the quantity of interest is the second vertical derivative ∂_z^2 of the vertical velocity component $u^z = u \cdot e_z$. The motivation for expressing the maximal regularity in the *borderline* spaces $L^1 \left(dt dx' \frac{1}{z(1-z)} dz \right)$ and $L_z^\infty(L_{t,x'}^1)$ comes from the nature of the right-hand-side $f = \text{Ra} T e_z - \frac{1}{\overline{P}_r} (u \cdot \nabla) u$ in the problem studied in [1]. Indeed, thanks to the no-slip boundary conditions, the convective nonlinearity is well controlled in the $L^1 \left(dt dx' \frac{1}{z(1-z)} dz \right)$ -norm, hence, a maximal regularity theory for the non-stationary Stokes equations with respect to this norm is required. The $L_z^\infty(L_{t,x'}^1)$ - norm arises for two unrelated reasons: It is needed to estimate the buoyancy term $T e_z$ driving the Navier-Stokes equations and it is the natural partner of $L^1 \left(dt dx' \frac{1}{z(1-z)} dz \right)$ in the maximal regularity estimate.

Aside from their application to the Rayleigh Bénard convection all the estimates in Theorem 1 might have an independent interest since they show the full extent of what one can obtain under the horizontal bandedness assumption only.

2 Maximal regularity in the strip

2.1 From the strip to the half space

Let us consider the non-stationary Stokes equations

$$\left\{ \begin{array}{ll} \partial_t u - \Delta u + \nabla p = f & \text{for } 0 < z < 1, \\ \nabla \cdot u = 0 & \text{for } 0 < z < 1, \\ u = 0 & \text{for } z \in \{0, 1\}, \\ u = 0 & \text{for } t = 0. \end{array} \right.$$

In order to prove the maximal regularity estimate in the strip we extend the problem (1) in the half space. By symmetry, it is enough to consider for the moment the extension to the upper half space.

Consider the localization $(\tilde{u}, \tilde{p}) := (\eta u, \eta p)$ where

$$\eta(z) \text{ is a cut-off function for } \left[0, \frac{1}{2}\right) \text{ in } [0, 1). \quad (5)$$

Extending (\tilde{u}, \tilde{p}) by zero they can be viewed as functions in the upper half space. The couple (\tilde{u}, \tilde{p}) satisfies

$$\begin{cases} \partial_t \tilde{u} - \Delta \tilde{u} + \nabla \tilde{p} = \tilde{f} & \text{for } z > 0, \\ \nabla \cdot \tilde{u} = \tilde{\rho} & \text{for } z > 0, \\ \tilde{u} = 0 & \text{for } z = 0, \\ \tilde{u} = 0 & \text{for } t = 0, \end{cases} \quad (6)$$

where

$$\tilde{f} := \eta f - 2(\partial_z \eta) \partial_z u - (\partial_z^2 \eta) u + (\partial_z \eta) p e_z, \quad \tilde{\rho} := (\partial_z \eta) u^z. \quad (7)$$

2.2 Maximal regularity in the upper half space

In the half space, taking advantages from the explicit representation of the solution via Green functions, we prove the regularity estimates which will be crucial in the proof of Theorem 1.

Proposition 1 (Maximal regularity in the upper half space).

Consider the non-stationary Stokes equations in the upper half-space

$$\begin{cases} \partial_t u - \Delta u + \nabla p = f & \text{for } z > 0, \\ \nabla \cdot u = \rho & \text{for } z > 0, \\ u = 0 & \text{for } z = 0, \\ u = 0 & \text{for } t = 0. \end{cases} \quad (8)$$

Suppose that f and ρ are horizontally band-limited, i.e

$$\mathcal{F}' f(k', z, t) = 0 \text{ unless } 1 \leq R|k'| \leq 4 \text{ where } R \in (0, \infty), \quad (9)$$

and

$$\mathcal{F}' \rho(k', z, t) = 0 \text{ unless } 1 \leq R|k'| \leq 4 \text{ where } R \in (0, \infty). \quad (10)$$

Then

$$\begin{aligned} & \| \partial_t u^z \|_{(0, \infty)} + \| \nabla^2 u^z \|_{(0, \infty)} + \| \nabla p \|_{(0, \infty)} + \| (\partial_t - \partial_z^2) u' \|_{(0, \infty)} + \| \nabla' \nabla u' \|_{(0, \infty)} \\ & \lesssim \| f \|_{(0, \infty)} + \| (-\Delta')^{-\frac{1}{2}} \partial_t \rho \|_{(0, \infty)} + \| (-\Delta')^{-\frac{1}{2}} \partial_z^2 \rho \|_{(0, \infty)} + \| \nabla \rho \|_{(0, \infty)}, \end{aligned}$$

where $\| \cdot \|_{(0, \infty)}$ denotes the norm

$$\| f \|_{(0, \infty)} := \| f \|_{R; (0, \infty)} \inf_{f=f_0+f_1} \left\{ \left\langle \sup_{0 < z < \infty} |f_0| \right\rangle + \left\langle \int_0^\infty |f_1| \frac{dz}{z} \right\rangle \right\}, \quad (11)$$

where f_0 and f_1 satisfy the bandedness assumption (9).

The first ingredient to establish Proposition 1 is a suitable representation of the solution operator $(f = (f', f^z), \rho) \rightarrow u = (u', u^z)$ of the Stokes equations with the no-slip boundary condition. In the case of no-slip boundary condition the Laplace operator has to be factorized as $\Delta = \partial_z^2 + \Delta' = (\partial_z + (-\Delta')^{\frac{1}{2}})(\partial_z - (-\Delta')^{\frac{1}{2}})$. In this way the solution operator to the Stokes equations with the no-slip boundary condition (8) can be written as the four-fold composition of solution operators to three more elementary boundary value problems:

- Backward fractional diffusion equation (12):

$$\begin{cases} (\partial_z - (-\Delta')^{\frac{1}{2}})\phi &= \nabla \cdot f - (\partial_t - \Delta)\rho & \text{for } z > 0, \\ \phi &\rightarrow 0 & \text{for } z \rightarrow \infty. \end{cases} \quad (12)$$

- Heat equation (13):

$$\begin{cases} (\partial_t - \Delta)v^z &= (-\Delta')^{\frac{1}{2}}(f^z - \phi) - \nabla' \cdot f' + (\partial_t - \Delta)\rho & \text{for } z > 0, \\ v^z &= 0 & \text{for } z = 0, \\ v^z &= 0 & \text{for } t = 0. \end{cases} \quad (13)$$

- Forward fractional diffusion equation (14):

$$\begin{cases} (\partial_z + (-\Delta')^{\frac{1}{2}})u^z &= v^z & \text{for } z > 0, \\ u^z &= 0 & \text{for } z = 0. \end{cases} \quad (14)$$

- Heat equation (15):

$$\begin{cases} (\partial_t - \Delta)v' &= (1 + \nabla'(-\Delta')^{-1}\nabla')f' & \text{for } z > 0, \\ v' &= 0 & \text{for } z = 0, \\ v' &= 0 & \text{for } t = 0. \end{cases} \quad (15)$$

Finally set

$$u' = v' - \nabla'(-\Delta')^{-1}(\rho - \partial_z u^z). \quad (16)$$

In order to prove the validity of the decomposition we need to argue that

$$(\partial_t - \Delta)u - f \text{ is irrotational ,}$$

which reduces to prove that

$$(\partial_t - \Delta)u' - f' \text{ is irrotational in } x'$$

and

$$\partial_z((\partial_t - \Delta)u' - f') = \nabla'((\partial_t - \Delta)u^z - f^z). \quad (17)$$

Let us consider for simplicity $\rho = 0$. The first statement follows easily from the definition. Indeed by definition (16) and equation (15),

$$(\partial_t - \Delta)u' - f' = \nabla'((-\Delta')^{-1}\nabla' \cdot f' + (-\Delta')^{-1}\partial_z u^z).$$

Let us now focus on (17), which by using (16) and (15) can be rewritten as

$$\partial_z \nabla'((-\Delta')^{-1}\nabla' \cdot f' + (-\Delta')^{-1}(\partial_t - \Delta)\partial_z u^z) = \nabla'((\partial_t - \Delta)u^z - f^z).$$

Because of the periodic boundary conditions in the horizontal direction, the latter is equivalent to

$$\partial_z(-\Delta')((-\Delta')^{-1}\nabla' \cdot f' + (-\Delta')^{-1}(\partial_t - \Delta)\partial_z u^z) = (-\Delta')((\partial_t - \Delta)u^z - f^z),$$

that, after factorizing $\Delta = (\partial_z - (-\Delta')^{\frac{1}{2}})(\partial_z + (-\Delta')^{\frac{1}{2}})$, turns into

$$(\partial_z - (-\Delta')^{\frac{1}{2}})(\partial_t - \Delta)(\partial_z + (-\Delta')^{\frac{1}{2}})u^z = (-\Delta')f^z - \partial_z \nabla' \cdot f'.$$

One can easily check that the identity holds true by applying (14), (13) and (12). The no-slip boundary condition is trivially satisfied, indeed by (14) we have $u^z = 0$ and $\partial_z u^z = 0$. The combination of (16) with $\partial_z u^z = 0$ gives $u' = 0$.

For each step of the decomposition of the Navier Stokes equations we will derive maximal regularity-type estimates. These are summed up in the following

Proposition 2.

1. Let ϕ, f, ρ satisfy the problem (12) and assume f, ρ are horizontally band-limited, i.e

$$\mathcal{F}' f(k', z, t) = 0 \text{ unless } 1 \leq R|k'| \leq 4$$

and

$$\mathcal{F}' \rho(k', z, t) = 0 \text{ unless } 1 \leq R|k'| \leq 4.$$

Then,

$$\|\phi\|_{(0,\infty)} \lesssim \|f\|_{(0,\infty)} + \|(-\Delta')^{-\frac{1}{2}}\partial_t\rho\|_{(0,\infty)} + \|\nabla\rho\|_{(0,\infty)}.$$

2. Let v^z, f, ϕ, ρ satisfy the problem (13) and assume f, ϕ, ρ are horizontally band-limited, i.e

$$\mathcal{F}'f(k', z, t) = 0 \text{ unless } 1 \leq R|k'| \leq 4,$$

$$\mathcal{F}'\phi(k', z, t) = 0 \text{ unless } 1 \leq R|k'| \leq 4$$

and

$$\mathcal{F}'\rho(k', z, t) = 0 \text{ unless } 1 \leq R|k'| \leq 4.$$

Then,

$$\begin{aligned} & \|\nabla v^z\|_{(0,\infty)} + \|(-\Delta)^{-\frac{1}{2}}(\partial_t - \partial_z^2)v^z\|_{(0,\infty)} \\ & \lesssim \|f\|_{(0,\infty)} + \|\phi\|_{(0,\infty)} + \|(-\Delta')^{-\frac{1}{2}}\partial_t\rho\|_{(0,\infty)} \\ & + \|(-\Delta)^{-\frac{1}{2}}\partial_z^2\rho\|_{(0,\infty)} + \|\nabla\rho\|_{(0,\infty)}. \end{aligned} \quad (18)$$

3. Let u^z, v^z satisfy the problem (14) and assume v^z is horizontally band-limited, i.e

$$\mathcal{F}'v^z(k', z, t) = 0 \text{ unless } 1 \leq R|k'| \leq 4.$$

Then,

$$\begin{aligned} & \|\partial_t u^z\|_{(0,\infty)} + \|\nabla^2 u^z\|_{(0,\infty)} + \|(-\Delta')^{-\frac{1}{2}}\partial_z(\partial_t - \partial_z^2)u^z\|_{(0,\infty)} \\ & \lesssim \|\nabla v^z\|_{(0,\infty)} + \|(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)v^z\|_{(0,\infty)}. \end{aligned} \quad (19)$$

4. Let v', f' , satisfy the problem (15) and assume f' is horizontally band-limited, i.e

$$\mathcal{F}'f(k', z, t) = 0 \text{ unless } 1 \leq R|k'| \leq 4.$$

Then,

$$\|\nabla'\nabla v'\|_{(0,\infty)} + \|(\partial_t - \partial_z^2)v'\|_{(0,\infty)} \lesssim \|f'\|_{(0,\infty)}. \quad (20)$$

2.3 Proof of Proposition 1

By an easy application of Proposition 2, we will now prove the maximal regularity estimate on the upper half space.

Proof of Proposition 1.

From Proposition 2 we have the following bound for the vertical component of the velocity u

$$\begin{aligned}
& \| \partial_t u^z \|_{(0,\infty)} + \| \nabla^2 u^z \|_{(0,\infty)} + \| (-\Delta')^{-\frac{1}{2}} \partial_z (\partial_t - \partial_z^2) u^z \|_{(0,\infty)} \\
(19) \quad & \lesssim \| \nabla v^z \|_{(0,\infty)} + \| (-\Delta')^{-\frac{1}{2}} (\partial_t - \partial_z^2) v^z \|_{(0,\infty)} \\
(18) \quad & \lesssim \| f \|_{(0,\infty)} + \| \phi \|_{(0,\infty)} + \| (-\Delta')^{-\frac{1}{2}} \partial_t \rho \|_{(0,\infty)} + \| (-\Delta)^{-\frac{1}{2}} \partial_z^2 \rho \|_{(0,\infty)} + \| \nabla \rho \|_{(0,\infty)} \\
(1) \quad & \lesssim \| f \|_{(0,\infty)} + \| (-\Delta')^{\frac{1}{2}} \partial_t \rho \|_{(0,\infty)} + \| (-\Delta)^{-\frac{1}{2}} \partial_z^2 \rho \|_{(0,\infty)} + \| \nabla \rho \|_{(0,\infty)}.
\end{aligned}$$

Instead for the horizontal components of the velocity u' we have

$$\begin{aligned}
& \| (\partial_t - \partial_z^2) u' \|_{(0,\infty)} + \| \nabla' \nabla u' \|_{(0,\infty)} \\
(16) \quad & \lesssim \| (\partial_t - \partial_z^2) v' \|_{(0,\infty)} + \| \nabla' \nabla v' \|_{(0,\infty)} \\
& + \| (-\Delta')^{-\frac{1}{2}} (\partial_t - \partial_z^2) \rho \|_{(0,\infty)} + \| \nabla \rho \|_{(0,\infty)} \\
& + \| (-\Delta')^{-\frac{1}{2}} \partial_z (\partial_t - \partial_z^2) u^z \|_{(0,\infty)} + \| \partial_z \nabla u^z \|_{(0,\infty)} \\
(18),(19),(20) \quad & \lesssim \| f \|_{(0,\infty)} + \| (-\Delta')^{-\frac{1}{2}} \partial_t \rho \|_{(0,\infty)} + \| (-\Delta)^{-\frac{1}{2}} \partial_z^2 \rho \|_{(0,\infty)} + \| \nabla \rho \|_{(0,\infty)}.
\end{aligned}$$

Summing up we obtain

$$\begin{aligned}
& \| \partial_t u^z \|_{(0,\infty)} + \| \nabla^2 u^z \|_{(0,\infty)} + \| (\partial_t - \partial_z^2) u' \|_{(0,\infty)} + \| \nabla' \nabla u' \|_{(0,\infty)} \\
& \lesssim \| f \|_{(0,\infty)} + \| (-\Delta')^{-\frac{1}{2}} \partial_t \rho \|_{(0,\infty)} + \| (-\Delta)^{-\frac{1}{2}} \partial_z^2 \rho \|_{(0,\infty)} + \| \nabla \rho \|_{(0,\infty)}.
\end{aligned} \tag{21}$$

The bound for the ∇p follows by equations (8) and applying (21). \square

2.4 Proof of Proposition 2

This section is devoted to the proof of Proposition 2, which rely on a series of Lemmas (Lemma 1, Lemma 2 and Lemma 3) that we state here and prove in Section 3.

The following Lemmas contain the basic maximal regularity estimates for the three auxiliary problems. These estimates, together with the bandedness assumption in the form of (82), (83) and (84) will be the main ingredients for the proof of Proposition 2.

Lemma 1.

Let u, f satisfy the problem

$$\begin{cases} (\partial_z - (-\Delta')^{\frac{1}{2}})u = f & \text{for } z > 0, \\ u \rightarrow 0 & \text{for } z \rightarrow \infty \end{cases} \quad (22)$$

and assume f to be horizontally band-limited, i.e

$$\mathcal{F}'f(k', z, t) = 0 \quad \text{unless} \quad 1 \leq R|k'| \leq 4.$$

Then,

$$\|\nabla u\|_{(0,\infty)} \lesssim \|f\|_{(0,\infty)}. \quad (23)$$

Lemma 2.

Let $u, f, g = g(x', t)$ satisfy the problem

$$\begin{cases} (\partial_z + (-\Delta')^{\frac{1}{2}})u = f & \text{for } z > 0, \\ u = g & \text{for } z = 0 \end{cases} \quad (24)$$

and define the constant extension $\tilde{g}(x', z, t) := g(x', t)$. Assume f and g to be horizontally band-limited, i.e

$$\mathcal{F}'f(k', z, t) = 0 \quad \text{unless} \quad 1 \leq R|k'| \leq 4$$

and

$$\mathcal{F}'g(k', z, t) = 0 \quad \text{unless} \quad 1 \leq R|k'| \leq 4.$$

Then

$$\|\nabla u\|_{(0,\infty)} \lesssim \|f\|_{(0,\infty)} + \|\nabla' \tilde{g}\|_{(0,\infty)}. \quad (25)$$

Remark 1. Clearly if $g = 0$ in Lemma 2, then we have

$$\|\nabla u\|_{(0,\infty)} \lesssim \|f\|_{(0,\infty)}. \quad (26)$$

Lemma 3.

Let u, f satisfy the problem

$$\begin{cases} (\partial_t - \Delta)u = f & \text{for } z > 0, \\ u = 0 & \text{for } z = 0, \\ u = 0 & \text{for } t = 0 \end{cases} \quad (27)$$

and assume f to be horizontally band-limited, i.e

$$\mathcal{F}'f(k', z, t) = 0 \quad \text{unless} \quad 1 \leq R|k'| \leq 4.$$

Then,

$$\|(\partial_t - \partial_z^2)u\|_{(0,\infty)} + \|\nabla' \nabla u\|_{(0,\infty)} \lesssim \|f\|_{(0,\infty)}. \quad (28)$$

Proof of Proposition 2.

1. Subtracting the quantity $(\partial_z - (-\Delta')^{\frac{1}{2}})(f^z + \partial_z \rho)$ from both sides of equation (12) and then multiplying the new equation by $(-\Delta)^{-\frac{1}{2}}$ we get

$$\begin{aligned} & (\partial_z - (-\Delta')^{\frac{1}{2}})(-\Delta')^{-\frac{1}{2}}(\phi - f^z - \partial_z \rho) \\ = & \nabla' \cdot (-\Delta')^{-\frac{1}{2}}f' + f^z - (-\Delta')^{-\frac{1}{2}}\partial_t \rho + \partial_z \rho - (-\Delta')^{\frac{1}{2}}\rho. \end{aligned}$$

From the basic estimate (23) we obtain

$$\begin{aligned} & \|\nabla'(-\Delta')^{-\frac{1}{2}}(\phi - f^z - \partial_z \rho)\|_{(0,\infty)} \lesssim \|\nabla' \cdot (-\Delta')^{-\frac{1}{2}}f'\|_{(0,\infty)} \\ + & \|f^z\|_{(0,\infty)} + \|(-\Delta')^{-\frac{1}{2}}\partial_t \rho\|_{(0,\infty)} + \|\partial_z \rho\|_{(0,\infty)} + \|(-\Delta')^{\frac{1}{2}}\rho\|_{(0,\infty)}. \end{aligned}$$

Thanks to the bandedness assumption in the form of (82) and (83) we have

$$\begin{aligned} & \|\phi - f^z - \partial_z \rho\|_{(0,\infty)} \\ \lesssim & \|f'\|_{(0,\infty)} + \|f^z\|_{(0,\infty)} + \|(-\Delta')^{-\frac{1}{2}}\partial_t \rho\|_{(0,\infty)} + \|\partial_z \rho\|_{(0,\infty)} + \|\nabla' \rho\|_{(0,\infty)} \end{aligned}$$

and from this we obtain easily the desired estimate (1).

2. After multiplying the equation (13) by $(-\Delta')^{-\frac{1}{2}}$, the application of (28) to $(-\Delta')^{-\frac{1}{2}}v^z$ yields

$$\begin{aligned} & \|(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)v^z\|_{(0,\infty)} + \|(-\Delta')^{-\frac{1}{2}}\nabla'\nabla v^z\|_{(0,\infty)} \\ & \lesssim \|f^z\|_{(0,\infty)} + \|\phi\|_{(0,\infty)} + \|\nabla' \cdot (-\Delta')^{-\frac{1}{2}}f'\|_{(0,\infty)} \\ & + \|(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)\rho\|_{(0,\infty)} + \|(-\Delta')^{\frac{1}{2}}\rho\|_{(0,\infty)}. \end{aligned}$$

The estimate (18) follows after observing (83) and applying the triangle inequality to the second to last term on the right hand side.

3. We need to estimate the the three terms on the right hand side of (19) separately. We start with the term $\nabla^2 u^z$: since $\|\nabla^2 u^z\|_{(0,\infty)} \leq \|\nabla'\nabla u^z\|_{(0,\infty)} + \|\partial_z^2 u^z\|_{(0,\infty)}$, we tackle the term $\nabla'\nabla u^z$ and $\partial_z^2 u^z$ separately. First multiply by ∇' the equation (14). An application of the estimate (26) to $\nabla' u^z$ yields

$$\|\nabla\nabla' u^z\|_{(0,\infty)} \lesssim \|\nabla' v^z\|_{(0,\infty)}. \quad (29)$$

Now multiplying the equation (14) by ∂_z^2

$$\partial_z^2 u^z = -(-\Delta')^{\frac{1}{2}}\partial_z u^z + \partial_z v^z = -\Delta' u^z - (-\Delta')^{\frac{1}{2}}v^z + \partial_z v^z \quad (30)$$

and using the bandedness assumption in the form (83) we have

$$\begin{aligned} \|\partial_z^2 u^z\|_{(0,\infty)} & \leq \|\nabla'^2 u^z\|_{(0,\infty)} + \|\nabla v^z\|_{(0,\infty)} \\ & \stackrel{(29)}{\leq} \|\nabla v^z\|_{(0,\infty)}. \end{aligned} \quad (31)$$

The second term of (19), i.e $(-\Delta')^{-\frac{1}{2}}\partial_z(\partial_t - \partial_z^2)u^z$, can be bounded in the following way: We multiply the equation (14) by $(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)$

$$\begin{cases} (\partial_z + (-\Delta')^{\frac{1}{2}})(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)u^z & = (-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)v^z & \text{for } z > 0, \\ (-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)u^z & = (-\Delta')^{-\frac{1}{2}}\partial_z v^z & \text{for } z = 0, \end{cases}$$

where we have used that at $z = 0$

$$(\partial_t - \partial_z^2)u^z = -\partial_z^2 u^z \stackrel{(30)}{=} \partial_z v^z.$$

Applying (25) to $(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)u^z$ and using the bandedness assumption in the form of (82),

$$\|\nabla(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)u^z\|_{(0,\infty)} \lesssim \|(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)v^z\|_{(0,\infty)} + \|\partial_z v^z\|_{(0,\infty)}. \quad (32)$$

Finally we can bound the last term of (19), i.e $\partial_t u^z$: We observe that $\partial_t u^z = (\partial_t - \partial_z^2)u^z + \partial_z^2 u^z$ thus

$$\|\partial_t u^z\|_{(0,\infty)} \leq \|(\partial_t - \partial_z^2)u^z\|_{(0,\infty)} + \|\partial_z^2 u^z\|_{(0,\infty)}. \quad (33)$$

For the first term in the right hand side of (33) we notice that

$$\begin{aligned} \|(\partial_t - \partial_z^2)u^z\|_{(0,\infty)} &\stackrel{(82)}{\leq} \|(-\Delta')^{-\frac{1}{2}}\nabla'(\partial_t - \partial_z^2)u^z\|_{(0,\infty)} \\ &\stackrel{(32)}{\lesssim} \|(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)v^z\|_{(0,\infty)} + \|\partial_z v^z\|_{(0,\infty)} \\ &\lesssim \|(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)v^z\|_{(0,\infty)} + \|\nabla v^z\|_{(0,\infty)}. \end{aligned}$$

The second term on the right hand side of (33) is bounded in (31). Thus we have the following bound for $\partial_t u$

$$\|\partial_t u^z\|_{(0,\infty)} \leq \|(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)v^z\|_{(0,\infty)} + \|\nabla v^z\|_{(0,\infty)}. \quad (34)$$

Putting together all the above we obtain the desired estimate.

4. From the defining equation (15), the basic estimate (28) and the bandedness assumption in form of (84), we get

$$\|(\partial_t - \partial_z^2)v'\|_{(0,\infty)} + \|\nabla'\nabla v'\|_{(0,\infty)} \lesssim \|f'\|_{(0,\infty)}.$$

□

2.5 Proof of Theorem 1

Let u, p, f be the solutions of the non-stationary Stokes equations in the strip $0 < z < 1$ (1). Then $\tilde{u} = \eta u, \tilde{p} = \eta p$ (with η defined in (5)) satisfy (6), namely

$$\begin{cases} \partial_t \tilde{u} - \Delta \tilde{u} + \nabla \tilde{p} &= \tilde{f} & \text{for } z > 0, \\ \nabla \cdot \tilde{u} &= \tilde{\rho} & \text{for } z > 0, \\ \tilde{u} &= 0 & \text{for } z = 0, \\ \tilde{u} &= 0 & \text{for } t = 0, \end{cases}$$

where

$$\tilde{f} := \eta f - 2(\partial_z \eta) \partial_z u - (\partial_z^2 \eta) u + (\partial_z \eta) p e_z, \quad \tilde{\rho} := (\partial_z \eta) u^z. \quad (35)$$

Since, by assumption f, ρ are horizontally band-limited, then also \tilde{f} and $\tilde{\rho}$ satisfy the horizontal bandedness assumption (9) and (10) respectively. We can therefore apply Proposition 1 to the upper half space problem (6) and get

$$\begin{aligned} & \|(\partial_t - \partial_z^2) \tilde{u}'\|_{(0,\infty)} + \|\nabla' \nabla \tilde{u}'\|_{(0,\infty)} + \|\partial_t \tilde{u}^z\|_{(0,\infty)} + \|\nabla^2 \tilde{u}^z\|_{(0,\infty)} + \|\nabla \tilde{p}\|_{(0,\infty)} \\ & \lesssim \|\tilde{f}\|_{(0,\infty)} + \|(-\Delta')^{-\frac{1}{2}} \partial_t \tilde{\rho}\|_{(0,\infty)} + \|(-\Delta')^{-\frac{1}{2}} \partial_z^2 \tilde{\rho}\|_{(0,\infty)} + \|\nabla \tilde{\rho}\|_{(0,\infty)}. \end{aligned}$$

By symmetry, we also have the same maximal regularity estimates in the lower half space. Indeed, let $\tilde{\tilde{u}}, \tilde{\tilde{p}}$ satisfy the equation

$$\begin{cases} \partial_t \tilde{\tilde{u}} - \Delta \tilde{\tilde{u}} + \nabla \tilde{\tilde{p}} &= \tilde{\tilde{f}} & \text{for } z < 1, \\ \nabla \cdot \tilde{\tilde{u}} &= \tilde{\tilde{\rho}} & \text{for } z < 1, \\ \tilde{\tilde{u}} &= 0 & \text{for } z = 1, \\ \tilde{\tilde{u}} &= 0 & \text{for } t = 0, \end{cases} \quad (36)$$

where

$$\tilde{\tilde{f}} := (1-\eta) f - 2(\partial_z(1-\eta)) \partial_z u - (\partial_z^2(1-\eta)) u + (\partial_z(1-\eta)) p e_z, \quad \tilde{\tilde{\rho}} := (\partial_z(1-\eta)) u^z. \quad (37)$$

Again by Proposition 2 we have

$$\begin{aligned} & \|(\partial_t - \partial_z^2) \tilde{\tilde{u}}'\|_{(-\infty,1)} + \|\nabla' \nabla \tilde{\tilde{u}}'\|_{(-\infty,1)} + \|\partial_t \tilde{\tilde{u}}^z\|_{(-\infty,1)} + \|\nabla^2 \tilde{\tilde{u}}^z\|_{(-\infty,1)} + \|\nabla \tilde{\tilde{p}}\|_{(-\infty,1)} \\ & \lesssim \|\tilde{\tilde{f}}\|_{(-\infty,1)} + \|(-\Delta')^{-\frac{1}{2}} \partial_t \tilde{\tilde{\rho}}\|_{(-\infty,1)} + \|(-\Delta')^{-\frac{1}{2}} \partial_z^2 \tilde{\tilde{\rho}}\|_{(-\infty,1)} + \|\nabla \tilde{\tilde{\rho}}\|_{(-\infty,1)}, \end{aligned}$$

where $\|\cdot\|_{(-\infty,1)}$ is the analogue of (11) (see Section (5) for notations). Since $u = \tilde{u} + \tilde{\tilde{u}}$ in the strip $[0, L)^{d-1} \times (0, 1)$, by the triangle inequality and using the maximal regularity estimates above, we get

$$\begin{aligned}
& \|(\partial_t - \partial_z^2)u'\|_{(0,1)} + \|\nabla' \nabla u'\|_{(0,1)} + \|\partial_t u^z\|_{(0,1)} + \|\nabla^2 u^z\|_{(0,1)} + \|\nabla p\|_{(0,1)} \\
& \lesssim \|(\partial_t - \partial_z^2)\tilde{u}'\|_{(0,\infty)} + \|(\partial_t - \partial_z^2)\tilde{\tilde{u}}'\|_{(-\infty,1)} + \|\nabla' \nabla \tilde{u}'\|_{(0,\infty)} + \|\nabla' \nabla \tilde{\tilde{u}}'\|_{(-\infty,1)} \\
& + \|\partial_t \tilde{u}^z\|_{(0,\infty)} + \|\partial_t \tilde{\tilde{u}}^z\|_{(-\infty,1)} + \|\nabla^2 \tilde{u}^z\|_{(0,\infty)} + \|\nabla^2 \tilde{\tilde{u}}^z\|_{(-\infty,1)} \\
& + \|\nabla \tilde{p}\|_{(0,\infty)} + \|\nabla \tilde{\tilde{p}}\|_{(-\infty,1)} \\
& \lesssim \|\tilde{f}\|_{(0,\infty)} + \|\tilde{\tilde{f}}\|_{(-\infty,1)} + \|(-\Delta')^{-\frac{1}{2}} \partial_t \tilde{\rho}\|_{(0,\infty)} + \|(-\Delta')^{-\frac{1}{2}} \partial_t \tilde{\tilde{\rho}}\|_{(-\infty,1)} \\
& + \|(-\Delta')^{-\frac{1}{2}} \partial_z^2 \tilde{\rho}\|_{(0,\infty)} + \|(-\Delta')^{-\frac{1}{2}} \partial_z^2 \tilde{\tilde{\rho}}\|_{(-\infty,1)} + \|\nabla \tilde{\rho}\|_{(0,\infty)} + \|\nabla \tilde{\tilde{\rho}}\|_{(-\infty,1)}.
\end{aligned}$$

By the definitions of \tilde{f} and $\tilde{\tilde{f}}$ we get

$$\|\tilde{f}\|_{(0,\infty)} + \|\tilde{\tilde{f}}\|_{(-\infty,1)} \lesssim \|f\|_{(0,1)} + \|\partial_z u\|_{(0,1)} + \|u\|_{(0,1)} + \|p\|_{(0,1)}$$

and similarly for $\tilde{\rho}$ and $\tilde{\tilde{\rho}}$ we have

$$\|\nabla \tilde{\rho}\|_{(0,\infty)} + \|\nabla \tilde{\tilde{\rho}}\|_{(-\infty,1)} \lesssim \|\nabla u\|_{(0,1)} + \|u\|_{(0,1)}$$

$$\|(-\Delta')^{-\frac{1}{2}} \partial_t \tilde{\rho}\|_{(0,\infty)} + \|(-\Delta')^{-\frac{1}{2}} \partial_t \tilde{\tilde{\rho}}\|_{(-\infty,1)} \lesssim \|(-\Delta')^{-\frac{1}{2}} \partial_t u\|_{(0,1)}$$

and

$$\begin{aligned}
& \|(-\Delta')^{-\frac{1}{2}} \partial_z^2 \tilde{\rho}\|_{(0,\infty)} + \|(-\Delta')^{-\frac{1}{2}} \partial_z^2 \tilde{\tilde{\rho}}\|_{(-\infty,1)} \\
& \lesssim \|(-\Delta')^{-\frac{1}{2}} u^z\|_{(0,1)} + \|(-\Delta')^{-\frac{1}{2}} \partial_z u^z\|_{(0,1)} + \|(-\Delta')^{-\frac{1}{2}} \partial_z^2 u^z\|_{(0,1)}.
\end{aligned}$$

Therefore, collecting the estimates, we have

$$\begin{aligned}
& \|(\partial_t - \partial_z^2)u'\|_{(0,1)} + \|\nabla' \nabla u'\|_{(0,1)} + \|\partial_t u^z\|_{(0,1)} + \|\nabla^2 u^z\|_{(0,1)} + \|\nabla p\|_{(0,1)} \\
& \lesssim \|f\|_{(0,1)} + \|p\|_{(0,1)} + \|\nabla u\|_{(0,1)} + \|u\|_{(0,1)} \\
& + \|(-\Delta')^{-\frac{1}{2}} \partial_t u\|_{(0,1)} + \|(-\Delta')^{-\frac{1}{2}} u^z\|_{(0,1)} + \|(-\Delta')^{-\frac{1}{2}} \partial_z u^z\|_{(0,1)} + \|(-\Delta')^{-\frac{1}{2}} \partial_z^2 u^z\|_{(0,1)}.
\end{aligned}$$

Incorporating the horizontal bandedness assumption we find

$$\begin{aligned}
\|\partial_z u\|_{(0,1)} &\leq R\|\nabla' \partial_z u\|_{(0,1)}, \\
\|u\|_{(0,1)} &\leq R^2\|(\nabla')^2 u\|_{(0,1)}, \\
\|p\|_{(0,1)} &\leq R\|\nabla' p\|_{(0,1)}, \\
\|\nabla u\|_{(0,1)} &\leq R\|\nabla' \nabla u\|_{(0,1)}, \\
\|(-\Delta')^{-\frac{1}{2}} \partial_t u\|_{(0,1)} &\leq R\|\partial_t u\|_{(0,1)}, \\
\|(-\Delta')^{-\frac{1}{2}} u^z\|_{(0,1)} &\leq R^3\|\nabla'^2 u^z\|_{(0,1)}, \\
\|(-\Delta')^{-\frac{1}{2}} \partial_z u^z\|_{(0,1)} &\leq R^2\|\nabla' \partial_z u^z\|_{(0,1)}, \\
\|(-\Delta')^{-\frac{1}{2}} \partial_z^2 u^z\|_{(0,1)} &\leq R\|\partial_z^2 u^z\|_{(0,1)}.
\end{aligned}$$

Thus, for $R < R_0$ where R_0 is sufficiently small, all the terms in the right hand side, except f can be absorbed into the left hand side and the conclusion follows.

3 Proof of main technical lemmas

Remark 2. *In the proof of Lemma 1, Lemma 2 and Lemma 3 we will derive inequalities between quantities where t is integrated between 0 and ∞ . From the proof it is clear that the same inequalities are true with t integrated between 0 and t_0 with constants that are not depending on t_0 . Therefore dividing by t_0 and taking $\limsup_{t_0 \rightarrow \infty}$ (see (92)) we shall obtain the desired estimates in terms of the interpolation norm (11).*

3.1 Proof of Lemma 1

Proof of Lemma 1.

In order to simplify the notations, in what follows we will omit the dependency of the functions from the time variable. It is enough to show

$$\|\nabla' u\|_{(0,\infty)} \lesssim \|f\|_{(0,\infty)},$$

since, by equation (22) $\partial_z u = (-\Delta')^{\frac{1}{2}} u + f$. We claim that, in order to prove (3.1), it is enough to show

$$\sup_z \langle |\nabla' u| \rangle' \lesssim \sup_z \langle |f| \rangle' \quad (38)$$

and

$$||\nabla' u||_{(0,\infty)} \lesssim \int \langle |f| \rangle' \frac{dz}{z}. \quad (39)$$

Indeed, by definition of the norm $||\cdot||_{(0,\infty)}$ (see (11)) if we select an arbitrary decomposition $\nabla' u = \nabla' u_1 + \nabla' u_2$, where u_1 and u_2 are solutions of the problem (22) with right hand sides f_1 and f_2 respectively, we have

$$\begin{aligned} ||\nabla' u||_{(0,\infty)} &\leq ||\nabla' u_1||_{(0,\infty)} + \sup_z \langle |\nabla' u_2| \rangle' \\ &\leq \int \langle |f_1| \rangle' \frac{dz}{z} + \sup_z \langle |f_2| \rangle'. \end{aligned}$$

Passing to the infimum over all the decompositions of f we obtain

$$||\nabla' u||_{(0,\infty)} \lesssim ||f||_{(0,\infty)}.$$

We recall that by Duhamel's principle we have the following representation

$$u(x', z) = \int_z^\infty u_{x', z_0}(z) dz_0, \quad (40)$$

where u_{z_0} is the harmonic extension of $f(\cdot, z_0)$ onto $\{z < z_0\}$, i.e it solves the boundary value problem

$$\begin{cases} (\partial_z - (-\Delta')^{\frac{1}{2}})u_{z_0} = 0 & \text{for } z < z_0, \\ u_{z_0} = f & \text{for } z = z_0. \end{cases} \quad (41)$$

Argument for (38):

Using the representation of the solution of (41) via the Poisson kernel, i.e

$$u_{z_0}(x', z) = \int \frac{z_0 - z}{(|x' - y'|^2 + (z_0 - z)^2)^{\frac{d}{2}}} f(x', z_0) dy'$$

we obtain the following bounds

$$\langle |\nabla' u_{z_0}(\cdot, z)| \rangle' \lesssim \begin{cases} \frac{1}{(z_0 - z)} & \langle |\nabla' f(\cdot, z_0)| \rangle', \\ \frac{1}{(z_0 - z)^2} & \langle |f(\cdot, z_0)| \rangle', \\ & \langle |\nabla' (-\Delta')^{-1} f(\cdot, z_0)| \rangle'. \end{cases} \quad (42)$$

By using the bandedness assumption in the form of (79) and (81), we have

$$\langle |\nabla' u_{z_0}(\cdot, z)| \rangle' \lesssim \min \left\{ \frac{1}{R}, \frac{R}{(z_0 - z)^2} \right\} \langle |f(\cdot, z_0)| \rangle',$$

hence

$$\begin{aligned} \langle |\nabla' u(\cdot, z)| \rangle' &\lesssim \int_z^\infty \min \left\{ \frac{1}{R}, \frac{R}{(z_0 - z)^2} \right\} \langle |f(\cdot, z_0)| \rangle' dz_0 \\ &\lesssim \sup_{z_0 \in (0, \infty)} \langle |f(\cdot, z_0)| \rangle' \int_z^\infty \min \left\{ \frac{1}{R}, \frac{R}{(z_0 - z)^2} \right\} dz_0 \\ &\lesssim \sup_{z_0 \in (0, \infty)} \langle |f(\cdot, z_0)| \rangle', \end{aligned}$$

which, passing to the supremum in z , implies (38).

From the above and applying Fubini's rule, we also have

$$\begin{aligned} \int_0^\infty \langle |\nabla' u(\cdot, z)| \rangle' dz &\leq \int_0^\infty \int_z^\infty \min \left\{ \frac{1}{R}, \frac{R}{(z_0 - z)^2} \right\} \langle |f(\cdot, z_0)| \rangle' dz_0 dz \quad (43) \\ &\leq \int_0^\infty \int_0^{z_0} \min \left\{ \frac{1}{R}, \frac{R}{(z_0 - z)^2} \right\} dz \langle |f(\cdot, z_0)| \rangle' dz_0 \\ &\lesssim \int_0^\infty \langle |f(\cdot, z)| \rangle' dz. \end{aligned}$$

Argument for (39):

Let us consider $\chi_{2H \leq z \leq 4H} f$ where $\chi_{2H \leq z \leq 4H}$ is the characteristic function on the interval $[2H, 4H]$ and let u_H be the solution to

$$(\partial_z - (-\Delta')^{\frac{1}{2}})u_H = \chi_{2H \leq z \leq 4H} f.$$

We claim

$$\sup_{z \leq H} \langle |\nabla' u_H| \rangle' \leq \int_0^\infty \langle |\chi_{2H \leq z \leq 4H} f| \rangle' \frac{dz}{z} \quad (44)$$

and

$$\int_H^\infty \langle |\nabla' u_H| \rangle' \frac{dz}{z} \leq \int_0^\infty \langle |\chi_{2H \leq z \leq 4H} f| \rangle' \frac{dz}{z}. \quad (45)$$

From estimate (44) and (45) the statement (39) easily follow. Indeed, choosing $H = 2^{n-1}$ and summing up over the dyadic intervals, we have

$$\begin{aligned}
\|\nabla' u\| &\leq \sum_{n \in \mathbb{Z}} \|\nabla' u_{2^{n-1}}\|_{(0, \infty)} \\
&\leq \sup_{z \leq 2^{n-1}} \langle |\nabla' u_{2^{n-1}}| \rangle' + \int_{2^{n-1}}^{\infty} \langle |\nabla' u_{2^{n-1}}| \rangle' \frac{dz}{z} \\
&\leq \sum_{n \in \mathbb{Z}} \int_0^{\infty} \langle |\chi_{2^n \leq z \leq 2^{n+1}} f| \rangle' \frac{dz}{z} \\
&= \int_0^{\infty} \langle |f| \rangle' \frac{dz}{z}.
\end{aligned}$$

Argument for (44): Fix $z \leq H$. Then, we have

$$\begin{aligned}
\langle |\nabla' u_H| \rangle' &\stackrel{(42)}{\leq} \int_z^{\infty} \frac{1}{(z_0 - z)} \langle |\chi_{2H \leq z \leq 4H} f(\cdot, z_0)| \rangle' dz_0 \\
&\lesssim \int_{2H}^{4H} \frac{1}{(z_0 - z)} \langle |\chi_{2H \leq z \leq 4H} f(\cdot, z_0)| \rangle' dz_0 \\
&\lesssim \frac{1}{H} \int_{2H}^{4H} \langle |\chi_{2H \leq z \leq 4H} f(\cdot, z_0)| \rangle' dz_0 \\
&\leq \int_{2H}^{\infty} \langle |\chi_{2H \leq z \leq 4H} f(\cdot, z_0)| \rangle' \frac{dz_0}{z_0} \\
&\leq \int_0^{\infty} \langle |\chi_{2H \leq z \leq 4H} f(\cdot, z_0)| \rangle' \frac{dz_0}{z_0}.
\end{aligned}$$

Taking the supremum over all z proves (44).

Argument for (45): For $z \geq H$ we have

$$\begin{aligned}
\int_H^{\infty} \langle |\nabla' u_H| \rangle' \frac{dz}{z} &\lesssim \frac{1}{H} \int_0^{\infty} \langle |\nabla' u_H| \rangle' dz \\
&\stackrel{(43)}{\lesssim} \frac{1}{H} \int_0^{\infty} \langle |\chi_{2H \leq z \leq 4H} f| \rangle' dz \\
&= \frac{1}{H} \int_{2H}^{4H} \langle |\chi_{2H \leq z \leq 4H} f| \rangle' dz \\
&\lesssim \int_0^{\infty} \langle |\chi_{2H \leq z \leq 4H} f| \rangle' \frac{dz}{z}.
\end{aligned}$$

□

3.2 Proof of Lemma 2

Proof of Lemma 2.

Let us first assume $g = 0$. It is enough to show

$$\sup_z \langle |\nabla' u| \rangle' \lesssim \sup_z \langle |f| \rangle' \quad (46)$$

and

$$\int_0^\infty \langle |\nabla' u| \rangle' \frac{dz}{z} \lesssim \int_0^\infty \langle |f| \rangle' \frac{dz}{z}. \quad (47)$$

Recall that by Duhamel's principle we have the following representation

$$u(z) = \int_0^z u_{z_0}(\cdot, z) dz_0, \quad (48)$$

where u_{z_0} is the harmonic extension of $f(z_0)$ onto $\{z > z_0\}$, i.e it solves the boundary value problem

$$\begin{cases} (\partial_z + (-\Delta')^{\frac{1}{2}})u_{z_0} = 0 & \text{for } z > z_0, \\ u_{z_0} = f & \text{for } z = z_0. \end{cases} \quad (49)$$

From the Poisson's kernel representation we learn that

$$\langle |\nabla' u_{z_0}(\cdot, z)| \rangle' \lesssim \begin{cases} \langle |\nabla' f(\cdot, z_0)| \rangle', \\ \frac{1}{(z-z_0)^2} \langle |\nabla' (-\Delta')^{-1} f(\cdot, z_0)| \rangle'. \end{cases}$$

Using the bandedness assumption in the form of (79) and (81)

$$\langle |\nabla' u_{z_0}(\cdot, z)| \rangle' \lesssim \min \left\{ \frac{1}{R}, \frac{R}{(z-z_0)^2} \right\} \langle |f(\cdot, z_0)| \rangle'$$

and observing (48), we obtain

$$\begin{aligned} \langle |\nabla' u(\cdot, z)| \rangle' &\lesssim \int_0^z \min \left\{ \frac{1}{R}, \frac{R}{(z-z_0)^2} \right\} \langle |f(\cdot, z_0)| \rangle' dz_0 \\ &\leq \sup_{z_0} \langle |f(\cdot, z_0)| \rangle' \int_0^z \min \left\{ \frac{1}{R}, \frac{R}{(z-z_0)^2} \right\} dz_0 \\ &\lesssim \sup_{z_0} \langle |f(\cdot, z_0)| \rangle'. \end{aligned} \quad (50)$$

Estimate (46) follows from (50) by passing to the supremum in z .
From the above (50), multiplying by the weight $\frac{1}{z}$ and observing that $z > z_0$ we have

$$\langle |\nabla' u(\cdot, z)| \rangle' \frac{1}{z} \lesssim \int_0^z \min \left\{ \frac{1}{R}, \frac{R}{(z - z_0)^2} \right\} \langle |f(\cdot, z_0)| \rangle' \frac{dz_0}{z_0}. \quad (51)$$

After integrating in $z \in (0, \infty)$ and applying Young's estimate we get (47).

Let's assume now the general case, with $g \neq 0$. We want to prove (25). Recall that by definition $\tilde{g}(x', z) := g(x')$ and consider $u - \tilde{g}$. By construction it satisfies

$$\begin{cases} (\partial_z + (-\Delta')^{-\frac{1}{2}})(u - \tilde{g}) &= f - (-\Delta')^{-\frac{1}{2}}g & \text{for } z > 0, \\ u - \tilde{g} &= 0 & \text{for } z = 0. \end{cases}$$

Using the first part of the proof of (26) and triangle inequality, we have

$$\|\nabla u\|_{(0, \infty)} \lesssim \|\nabla \tilde{g}\|_{(0, \infty)} + \|f\|_{(0, \infty)} + \|(-\Delta')^{\frac{1}{2}} \tilde{g}\|_{(0, \infty)}.$$

Therefore by the bandedness assumption in the form of (83) we can conclude (25). \square

3.3 Proof of Lemma 3

Proof of Lemma 3.

We will show that, for the non-homogeneous heat equation with Dirichlet boundary condition

$$\begin{cases} (\partial_t - \Delta)u &= f & \text{for } z > 0, \\ u &= 0 & \text{for } z = 0, \\ u &= 0 & \text{for } t = 0, \end{cases} \quad (52)$$

we have the following estimates

$$\left\langle \int |(\partial_t - \partial_z^2)u(\cdot, z, \cdot)| \frac{dz}{z} \right\rangle + \left\langle \int |\nabla'^2 u(\cdot, z, \cdot)| \frac{dz}{z} \right\rangle \lesssim \left\langle \int |f(\cdot, z, \cdot)| \frac{dz}{z} \right\rangle, \quad (53)$$

$$\langle |\nabla' \partial_z u(\cdot, z, \cdot)|_{z=0} \rangle \lesssim \left\langle \int |f(\cdot, z, \cdot)| \frac{dz}{z} \right\rangle, \quad (54)$$

$$\left\langle \sup_z |\nabla'^2 u(\cdot, z, \cdot)| \right\rangle \lesssim \left\langle \sup_z |f(\cdot, z, \cdot)| \right\rangle, \quad (55)$$

$$\left\langle \sup_z |\nabla' \partial_z u(\cdot, z, \cdot)| \right\rangle \lesssim \left\langle \sup_z |f(\cdot, z, \cdot)| \right\rangle. \quad (56)$$

In order to bound the off-diagonal components of the Hessian, we consider the decomposition

$$u = u_N + u_C, \quad (57)$$

where u_N solves

$$\begin{cases} (\partial_t - \Delta)u_N = f & \text{for } z > 0, \\ \partial_z u_N = 0 & \text{for } z = 0, \\ u_N = 0 & \text{for } t = 0, \end{cases} \quad (58)$$

and u_C solves

$$\begin{cases} (\partial_t - \Delta)u_C = 0 & \text{for } z > 0, \\ \partial_z u_C = \partial_z u & \text{for } z = 0, \\ u_C = 0 & \text{for } t = 0. \end{cases} \quad (59)$$

The splitting (57) is valid by the uniqueness of the Neumann problem. For the auxiliary problems (58) and (59) we have the following bounds

$$\left\langle \int |\nabla' \partial_z u_N(\cdot, z, \cdot)| \frac{dz}{z} \right\rangle \lesssim \left\langle \int |f(\cdot, z, \cdot)| \frac{dz}{z} \right\rangle, \quad (60)$$

$$\left\langle \sup_z |\nabla' \partial_z u_C(\cdot, z, \cdot)| \right\rangle \lesssim \langle |\nabla' \partial_z u(\cdot, z, \cdot)|_{z=0} \rangle. \quad (61)$$

We claim that estimates (53), (54), (55), (56), (60) and (61) yield (28).

Let us first consider the bound for ∇'^2 . Consider $u = u_1 + u_2$, where u_1 and u_2 satisfy (52) with right hand side f_1 and f_2 respectively. We have

$$\begin{aligned} \|\nabla'^2 u\|_{(0,\infty)} &\lesssim \left\langle \sup_z |\nabla'^2 u_1| \right\rangle + \left\langle \int |\nabla'^2 u_2| \frac{dz}{z} \right\rangle \\ &\stackrel{(53)\&(55)}{\lesssim} \left\langle \sup_z |f_1| \right\rangle + \left\langle \int |f_2| \frac{dz}{z} \right\rangle, \end{aligned}$$

which implies, upon taking infimum over all decompositions $f = f_1 + f_2$

$$\|\nabla'^2 u\|_{(0,\infty)} \lesssim \|f\|_{(0,\infty)}. \quad (62)$$

We now consider a further decomposition of u_2 , i.e $u_2 = u_{2C} + u_{2N}$ where u_{2C} satisfies (59) and u_{2N} satisfies (58). Therefore $u = u_1 + u_{2C} + u_{2N}$ and we can bound the off-diagonal components of the Hessian

$$\begin{aligned} \|\nabla' \partial_z u\|_{(0,\infty)} &\lesssim \left\langle \sup_z |\nabla' \partial_z u_1| \right\rangle + \left\langle \sup_z |\nabla' \partial_z u_{2C}| \right\rangle + \left\langle \int |\nabla' \partial_z u_{2N}| \frac{dz}{z} \right\rangle \\ &\stackrel{(54),(61),(60)\&(56)}{\lesssim} \left\langle \sup_z |f_1| \right\rangle + \left\langle \int |f_2| \frac{dz}{z} \right\rangle. \end{aligned}$$

From the last inequality, passing to the infimum over all the possible decompositions of f we get

$$\|\nabla' \partial_z u\|_{(0,\infty)} \lesssim \|f\|_{(0,\infty)}. \quad (63)$$

On one hand estimate (62) and (63) imply

$$\|\nabla \nabla' u\|_{(0,\infty)} \lesssim \|\nabla'^2 u\|_{(0,\infty)} + \|\nabla' \partial_z u\|_{(0,\infty)},$$

on the other hand equation (27) and estimate (62) yield

$$\|(\partial_t - \partial_z^2)u\|_{(0,\infty)} \lesssim \|f\|_{(0,\infty)}.$$

Argument for (53)

Let u be a solution of problem of (52). Keeping in mind Remark (2) it is enough to show

$$\int_0^\infty \int_0^\infty \langle |\nabla'^2 u| \rangle' \frac{dz}{z} dt \lesssim \int_0^\infty \int_0^\infty \langle |f| \rangle' \frac{dz}{z} dt.$$

By the Duhamel's principle we have

$$u(x', z, t) = \int_{s=0}^t u_s(x', z, t) ds, \quad (64)$$

where u_s is the solution to the homogeneous, initial value problem

$$\begin{cases} (\partial_t - \Delta)u_s = 0 & \text{for } z > 0, t > s, \\ u_s = 0 & \text{for } z = 0, t > s, \\ u_s = f & \text{for } z > 0, t = s. \end{cases} \quad (65)$$

Extending u and f to the whole space by odd reflection¹, we are left to study the problem

$$\begin{cases} (\partial_t - \Delta)u_s = 0 & \text{for } z \in \mathbb{R}, t > s, \\ u_s = f & \text{for } z \in \mathbb{R}, t = s, \end{cases}$$

¹with abuse of notation we will call again u and f these extensions.

the solution of which can be represented via heat kernel as

$$\begin{aligned} u_s(x', z, t) &= \int_{\mathbb{R}} \Gamma(\cdot, z - \tilde{z}, t - s) *_{x'} f(\cdot, \tilde{z}, s) d\tilde{z} \\ &= \int_0^\infty [\Gamma(\cdot, z - \tilde{z}, t - s) - \Gamma(\cdot, z + \tilde{z}, t - s)] *_{x'} f(\cdot, \tilde{z}, s) d\tilde{z}. \end{aligned} \quad (66)$$

The application of ∇'^2 to the representation above yields

$$\begin{aligned} &\nabla'^2 u_s(x', z, t) \\ &= \left\{ \int_0^\infty \int_{\mathbb{R}^{d-1}} \nabla' \Gamma_{d-1}(x' - \tilde{x}', t - s) (\Gamma_1(z - \tilde{z}, t - s) - \Gamma_1(z + \tilde{z}, t - s)) \nabla' f(\tilde{x}', \tilde{z}, s) d\tilde{x}' d\tilde{z}, \right. \\ &\quad \left. \int_0^\infty \int_{\mathbb{R}^{d-1}} \nabla'^3 \Gamma_{d-1}(x' - \tilde{x}', t - s) (\Gamma_1(z - \tilde{z}, t - s) - \Gamma_1(z + \tilde{z}, t - s)) (-\Delta')^{-1} \nabla' f(\tilde{x}', \tilde{z}, s) d\tilde{x}' d\tilde{z} \right\}. \end{aligned}$$

Averaging in the horizontal direction we obtain, on the one hand

$$\begin{aligned} &\langle |\nabla'^2 u_s(\cdot, z, t)| \rangle' \\ &\lesssim \int_0^\infty \langle |\nabla' \Gamma_{d-1}(\cdot, t - s)| \rangle' |\Gamma_1(z - \tilde{z}, t - s) - \Gamma_1(z + \tilde{z}, t - s)| \langle |\nabla' f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z} \\ &\stackrel{(87) \& (81)}{\lesssim} \int_0^\infty \frac{1}{(t-s)^{\frac{1}{2}}} |\Gamma_1(z - \tilde{z}, t - s) - \Gamma_1(z + \tilde{z}, t - s)| \frac{1}{R} \langle |f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z} \end{aligned}$$

and, on the other hand

$$\begin{aligned} &\langle |\nabla'^2 u_s(\cdot, z, t)| \rangle' \\ &\lesssim \int_0^\infty \langle |\nabla'^3 \Gamma_{d-1}(\cdot, t - s)| \rangle' |\Gamma_1(z - \tilde{z}, t - s) - \Gamma_1(z + \tilde{z}, t - s)| \langle |(-\Delta')^{-1} \nabla' f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z} \\ &\stackrel{(87) \& (79)}{\lesssim} \int_0^\infty |\Gamma_1(z - \tilde{z}, t - s) - \Gamma_1(z + \tilde{z}, t - s)| \frac{1}{(t-s)^{\frac{3}{2}}} R \langle |f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z}. \end{aligned}$$

Multiplying by the weight $\frac{1}{z}$ and integrating in $z \in (0, \infty)$ we get

$$\int_0^\infty \langle |\nabla'^2 u_s(\cdot, t)| \rangle' \frac{dz}{z} \lesssim \left(\sup_{\tilde{z}} \int_0^\infty K_{t-s}(z, \tilde{z}) dz \right) \left\{ \frac{1}{(t-s)^{\frac{1}{2}}} \frac{1}{R} \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' \frac{d\tilde{z}}{\tilde{z}}, \right. \\ \left. \frac{R}{(t-s)^{\frac{3}{2}}} \int_0^\infty \langle |f(x', \tilde{z}, s)| \rangle' \frac{d\tilde{z}}{\tilde{z}}, \right.$$

where we called $K_{t-s}(z, \tilde{z}) = \frac{\tilde{z}}{z} |\Gamma_1(z - \tilde{z}, t - s) - \Gamma_1(z + \tilde{z}, t - s)|$.

From Lemma 5 we infer

$$\sup_{\tilde{z}} \int_0^\infty K_{t-s}(z, \tilde{z}) dz \stackrel{(86)}{\lesssim} \int_{\mathbb{R}} |\Gamma_1(z, t - s)| dz + \sup_{z \in \mathbb{R}} (z^2 |\partial_z \Gamma_1(z, t - s)|) \stackrel{(88) \& (91)}{\lesssim} 1$$

and therefore we have

$$\int_0^\infty \langle |\nabla'^2 u_s(\cdot, z, t)| \rangle' \frac{dz}{z} \lesssim \left\{ \frac{1}{(t-s)^{\frac{1}{2}}} \frac{1}{R} \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' \frac{d\tilde{z}}{\tilde{z}}, \right. \\ \left. \frac{1}{(t-s)^{\frac{3}{2}}} R \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' \frac{d\tilde{z}}{\tilde{z}} \right\}.$$

Finally, inserting the previous estimate into the Duhamel formula (64) and integrating in time we get

$$\begin{aligned}
& \int_0^\infty \langle |\nabla'^2 u(\cdot, z, t)| \rangle' \frac{dz}{z} dt \\
(64) \quad & \lesssim \int_0^\infty \int_0^t \langle |\nabla'^2 u_s(\cdot, z, t)| \rangle' \frac{dz}{z} ds dt \\
& \lesssim \int_0^\infty \int_s^\infty \min \left\{ \frac{1}{R(t-s)^{\frac{1}{2}}}, \frac{R}{(t-s)^{\frac{3}{2}}} \right\} \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' \frac{d\tilde{z}}{\tilde{z}} dt ds \\
& \lesssim \int_0^\infty \int_s^\infty \min \left\{ \frac{1}{R(t-s)^{\frac{1}{2}}}, \frac{R}{(t-s)^{\frac{3}{2}}} \right\} dt \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' \frac{d\tilde{z}}{\tilde{z}} ds \quad (67) \\
& \lesssim \int_0^\infty \int_0^\infty \min \left\{ \frac{1}{R\tau^{\frac{1}{2}}}, \frac{R}{\tau^{\frac{3}{2}}} \right\} d\tau \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' \frac{d\tilde{z}}{\tilde{z}} ds, \quad (68) \\
& \lesssim \int_0^\infty \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' \frac{d\tilde{z}}{\tilde{z}} ds,
\end{aligned}$$

where in the second to last inequality we used

$$\int_0^\infty \min \left\{ \frac{1}{R\tau^{\frac{1}{2}}}, \frac{R}{\tau^{\frac{3}{2}}} \right\} d\tau \lesssim 1. \quad (69)$$

Argument for (54):

Let u be a solution of problem of (52). Recall that we need to prove

$$\int_0^\infty \langle |\nabla' \partial_z u|_{z=0}(\cdot, z, t)| \rangle' dt \lesssim \int_0^\infty \int_0^\infty \langle |f(\cdot, z, t)| \rangle' dt \frac{dz}{z}. \quad (70)$$

The solution of the equation (65) extended to the whole space by odd reflection can be represented by (66) (see argument for (53)). Therefore

$$\begin{aligned}
& \nabla' \partial_z u_s(x', z, t)|_{z=0} \\
= & \begin{cases} -2 \int_{\mathbb{R}^{d-1}} \int_0^\infty \Gamma_{d-1}(x' - \tilde{x}', t-s) \partial_z \Gamma_1(\tilde{z}, t-s) \nabla' f(\tilde{x}', \tilde{z}, s) d\tilde{x}' d\tilde{z}, \\ -2 \int_{\mathbb{R}^{d-1}} \int_0^\infty \nabla' \Gamma_{d-1}(x' - \tilde{x}', t-s) \partial_z \Gamma_1(\tilde{z}, t-s) \nabla' (-\Delta')^{-1} \nabla' f(\tilde{x}', \tilde{z}, s) d\tilde{x}' d\tilde{z}. \end{cases}
\end{aligned}$$

Taking the horizontal average we get, on the one hand

$$\begin{aligned}
& \langle |\nabla' \partial_z u_s(\cdot, z, t)|_{z=0} \rangle' \\
& \lesssim \int_0^\infty \langle |\Gamma_{d-1}(\cdot, t-s)| \rangle' |\partial_z \Gamma_1(\tilde{z}, t-s)| \langle |\nabla' f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z} \\
& \stackrel{(87)}{\lesssim} \int_0^\infty |\partial_z \Gamma_1(\tilde{z}, t-s)| \langle |\nabla' f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z} \\
& \stackrel{(81)}{\lesssim} \frac{1}{R} \int_0^\infty |\partial_z \Gamma_1(\tilde{z}, t-s)| \langle |f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z} \\
& \lesssim \frac{1}{R} \sup_{\tilde{z}} |\tilde{z} \partial_z \Gamma_1(\tilde{z}, t-s)| \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' \frac{d\tilde{z}}{\tilde{z}}
\end{aligned}$$

and on the other hand

$$\begin{aligned}
& \langle |\nabla' \partial_z u_s(\cdot, z, t)|_{z=0} \rangle' \\
& \lesssim \int_0^\infty \langle |(\nabla')^2 \Gamma_{d-1}(\cdot, t-s)| \rangle' |\partial_z \Gamma_1(\tilde{z}, t-s)| \langle |(-\Delta')^{-1} \nabla' f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z} \\
& \stackrel{(87)}{\lesssim} \frac{1}{(t-s)} \int_0^\infty |\partial_z \Gamma_1(\tilde{z}, t-s)| \langle |(-\Delta')^{-1} \nabla' f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z} \\
& \stackrel{(79)}{\lesssim} \frac{R}{(t-s)} \int_0^\infty |\partial_z \Gamma_1(\tilde{z}, t-s)| \langle |f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z} \\
& \lesssim \frac{R}{(t-s)} \sup_{\tilde{z}} |\tilde{z} \partial_z \Gamma_1(\tilde{z}, t-s)| \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' \frac{d\tilde{z}}{\tilde{z}}.
\end{aligned}$$

Using the estimate (90) we get

$$\langle |\nabla' \partial_z u_s(x', z, t)|_{z=0} \rangle' \lesssim \begin{cases} \frac{1}{(t-s)^{1/2} R} \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' \frac{d\tilde{z}}{\tilde{z}}, \\ \frac{R}{(t-s)^{3/2}} \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' \frac{d\tilde{z}}{\tilde{z}}. \end{cases}$$

Finally, inserting into Duhamel's formula and integrating in time we have

$$\begin{aligned}
& \int_0^\infty \langle |\nabla' \partial_z u(\cdot, z, t)|_{z=0} \rangle' dt \\
(64) \quad & \lesssim \int_0^\infty \int_0^t \langle |\nabla' \partial_z u_s(\cdot, z, t)|_{z=0} \rangle' ds dt \\
& \lesssim \int_0^\infty \int_s^\infty \min\left\{ \frac{1}{R(t-s)^{\frac{1}{2}}}, \frac{R}{(t-s)^{\frac{3}{2}}} \right\} \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' \frac{d\tilde{z}}{\tilde{z}} dt ds \\
(67)\&(68) \quad \lesssim \int_0^\infty \int_0^\infty \langle |f(x', z, s)| \rangle' \frac{d\tilde{z}}{\tilde{z}} ds.
\end{aligned}$$

Argument for (55):

Let u be the solution of problem (52). We recall that we want to prove

$$\sup_z \int_0^\infty \langle |\nabla'^2 u(\cdot, z, t)| \rangle' dt \lesssim \sup_z \int_0^\infty \langle |f(\cdot, z, t)| \rangle' dt. \quad (71)$$

The solution of equation (65) extended to the whole space can be represented by (66) (see argument for (53)). Therefore applying ∇'^2 to (66) and considering the horizontal average we have, on the one hand

$$\begin{aligned}
& \langle |\nabla'^2 u_s(\cdot, z, t)| \rangle' \\
& \lesssim \int_{\mathbb{R}} \langle |\nabla' \Gamma_{d-1}(\cdot, t-s)| \rangle' |\Gamma_1(z - \tilde{z}, t-s)| \langle |\nabla' f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z} \\
(87)\&(81) \quad \lesssim \int_{\mathbb{R}} \frac{1}{(t-s)^{\frac{1}{2}}} |\Gamma_1(z - \tilde{z}, t-s)| \frac{1}{R} \langle |f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z}
\end{aligned}$$

and on the other hand

$$\begin{aligned}
& \langle |\nabla'^2 u_s(\cdot, z, t)| \rangle' \\
& \lesssim \int_{\mathbb{R}} \langle |\nabla'^3 \Gamma_{d-1}(\cdot, t-s)| \rangle' |\Gamma_1(z - \tilde{z}, t-s)| \langle |(-\Delta')^{-1} \nabla' f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z} \\
(87)\&(79) \quad \lesssim \int_{\mathbb{R}} \frac{1}{(t-s)^{\frac{3}{2}}} |\Gamma_1(z - \tilde{z}, t-s)| R \langle |f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z}.
\end{aligned}$$

Inserting the above estimates in the Duhamel's formula (64), we have

$$\begin{aligned}
& \int_0^\infty \int_0^t \langle |\nabla'^2 u_s(z, \cdot)| \rangle' ds dt \\
& \lesssim \int_0^\infty \int_s^\infty \min \left\{ \frac{1}{R(t-s)^{\frac{1}{2}}}, \frac{R}{(t-s)^{\frac{3}{2}}} \right\} \int_{\mathbb{R}} |\Gamma_1(z - \tilde{z}, t-s)| \langle |f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z} ds dt \\
& \lesssim \int_{\mathbb{R}} \left(\int_0^\infty \min \left\{ \frac{1}{R\tau^{\frac{1}{2}}}, \frac{R}{\tau^{\frac{3}{2}}} \right\} |\Gamma_1(z - \tilde{z}, \tau)| d\tau \right) \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' ds d\tilde{z} \\
& \lesssim \sup_{\tilde{z}} \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' ds \int_{\mathbb{R}} \int_0^\infty \min \left\{ \frac{1}{R\tau^{\frac{1}{2}}}, \frac{R}{\tau^{\frac{3}{2}}} \right\} |\Gamma_1(z - \tilde{z}, \tau)| d\tau d\tilde{z} \\
& \stackrel{(88)}{\lesssim} \sup_{\tilde{z}} \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' ds \int_0^\infty \min \left\{ \frac{1}{R\tau^{\frac{1}{2}}}, \frac{R}{\tau^{\frac{3}{2}}} \right\} d\tau \int_{\mathbb{R}} |\Gamma_1(z - \tilde{z}, \tau)| d\tilde{z} \\
& \stackrel{(69)}{\lesssim} \sup_{\tilde{z}} \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' ds.
\end{aligned}$$

Taking the supremum in z we obtain the desired estimate.

Argument for (56):

Let u be the solution of problem (52). We claim

$$\sup_z \int_0^\infty \langle |\nabla' \partial_z u| \rangle' dt \lesssim \sup_z \int_0^\infty \langle |f| \rangle' dt. \quad (72)$$

The solution of the equation (65) extended to the whole space can be represented by (see argument for (53))

$$u_s(x', z, t) = \int_{\mathbb{R}} \Gamma(\cdot, z - \tilde{z}, t-s) *_{x'} f(\cdot, \tilde{z}, s) d\tilde{z}.$$

Applying $\nabla' \partial_z$ and considering the horizontal average we obtain, on the one hand

$$\begin{aligned}
& \langle |\nabla' \partial_z u_s(\cdot, z, t)| \rangle' \\
& \lesssim \int_{\mathbb{R}} \langle |\Gamma_{d-1}(\cdot, t-s)| \rangle' |\partial_z \Gamma_1(z - \tilde{z}, t-s)| \langle |\nabla' f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z} \\
& \stackrel{(81)}{\lesssim} \int_{\mathbb{R}} |\partial_z \Gamma_1(z - \tilde{z}, t-s)| \frac{1}{R} \langle |f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z}
\end{aligned}$$

and, on the other hand

$$\begin{aligned}
& \langle |\nabla' \partial_z u_s(\cdot, z, t)| \rangle' \\
& \lesssim \int_{\mathbb{R}} \langle |\nabla'^2 \Gamma_{d-1}(\cdot, t-s)| \rangle' |\partial_z \Gamma_1(z - \tilde{z}, t-s)| \langle |(-\Delta')^{-1} \nabla' f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z} \\
(79) \quad & \lesssim \int_{\mathbb{R}} \frac{1}{(t-s)} |\partial_z \Gamma_1(z - \tilde{z}, t-s)| R \langle |f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z}.
\end{aligned}$$

Inserting the above estimates in the Duhamel's formula (64), we have

$$\begin{aligned}
& \int_0^\infty \int_0^t \langle |\nabla' \partial_z u_s(z, \cdot)| \rangle' ds dt \\
& \lesssim \int_0^\infty \int_s^\infty \min \left\{ \frac{1}{R}, \frac{R}{(t-s)} \right\} \int_{\mathbb{R}} |\partial_z \Gamma_1(z - \tilde{z}, t-s)| \langle |f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z} dt ds \\
& \lesssim \int_{\mathbb{R}} \left(\int_0^\infty \min \left\{ \frac{1}{R}, \frac{R}{\tau} \right\} |\partial_z \Gamma_1(z - \tilde{z}, \tau)| d\tau \right) \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' ds d\tilde{z} \\
& \lesssim \sup_{\tilde{z}} \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' ds \int_{\mathbb{R}} \int_0^\infty \min \left\{ \frac{1}{R}, \frac{R}{\tau} \right\} |\partial_z \Gamma_1(z - \tilde{z}, \tau)| d\tau d\tilde{z} \\
(88) \quad & \lesssim \sup_{\tilde{z}} \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' ds \int_0^\infty \min \left\{ \frac{1}{R\tau^{\frac{1}{2}}}, \frac{R}{\tau^{\frac{3}{2}}} \right\} d\tau \\
(69) \quad & \lesssim \sup_{\tilde{z}} \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' ds.
\end{aligned}$$

Taking the supremum in z we obtain the desired estimate.

Argument for (60)

We recall that we want to show

$$\int_0^\infty \int_0^\infty \langle |\nabla' \partial_z u_N| \rangle' \frac{dz}{z} dt \lesssim \int_0^\infty \int_0^\infty \langle |f| \rangle' \frac{dz}{z} dt,$$

where u_N be the solution to the non-homogeneous heat equation with Neumann boundary conditions (58). By the Duhamel's principle we have

$$u_N(x', z, t) = \int_{s=0}^t u_{N_s}(x', z, t) ds,$$

where u_{N_s} is solution to

$$\begin{cases} (\partial_t - \Delta) u_{N_s} &= 0 & \text{for } z > 0, t > s, \\ \partial_z u_{N_s} &= 0 & \text{for } z = 0, t > s, \\ u_{N_s} &= f & \text{for } z > 0, t = s, \end{cases}$$

is the solution of problem (52). Extending this equation to the whole space by even reflection ², we are left to study the problem

$$\begin{cases} (\partial_t - \Delta)u_{N_s} = 0 & \text{for } z \in \mathbb{R}, t > s, \\ u_{N_s} = f & \text{for } t = s, \end{cases}$$

the solution of which can be represented via heat kernel as

$$\begin{aligned} u_{N_s}(x', z, t) &= \int_{\mathbb{R}} \Gamma(\cdot, z - \tilde{z}, t - s) *_{x'} f(\cdot, \tilde{z}, s) d\tilde{z} \\ &= \int_0^\infty [\Gamma(\cdot, \tilde{z} + z, t - s) + \Gamma(\cdot, \tilde{z} - z, t - s)] *_{x'} f(\cdot, \tilde{z}, s) d\tilde{z}. \end{aligned}$$

Applying $\nabla' \partial_z$ to the representation above

$$\begin{aligned} &\nabla' \partial_z u_{N_s}(x', z, t) \\ &= \begin{cases} \int_0^\infty \int_{\mathbb{R}^{d-1}} \Gamma_{d-1}(x' - \tilde{x}', t - s) (\partial_z \Gamma_1(\tilde{z} + z, t - s) - \partial_z \Gamma_1(\tilde{z} - z, t - s)) \nabla' f(\tilde{x}', \tilde{z}, s) d\tilde{x}' d\tilde{z}, \\ \int_0^\infty \int_{\mathbb{R}^{d-1}} \nabla'^2 \Gamma_{d-1}(x' - \tilde{x}', t - s) (\partial_z \Gamma_1(\tilde{z} + z, t - s) - \partial_z \Gamma_1(\tilde{z} - z, t - s)) (-\Delta')^{-1} \nabla' f(\tilde{x}', \tilde{z}, s) d\tilde{x}' d\tilde{z} \end{cases} \end{aligned}$$

and averaging in the horizontal direction we obtain, on the one hand

$$\begin{aligned} &\langle |\nabla' \partial_z u_{N_s}(\cdot, z, t)| \rangle' \\ &\lesssim \int_0^\infty \langle |\Gamma_{d-1}(\cdot, t - s)| \rangle' |\partial_z \Gamma_1(\tilde{z} + z, t - s) - \partial_z \Gamma_1(\tilde{z} - z, t - s)| \langle |\nabla' f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z} \\ (87) \&(81) \lesssim \frac{1}{R} \int_0^\infty |\partial_z \Gamma_1(\tilde{z} + z, t - s) - \partial_z \Gamma_1(\tilde{z} - z, t - s)| \langle |f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z} \end{aligned}$$

and, on the other hand

$$\begin{aligned} &\langle |\nabla' \partial_z u_{N_s}(\cdot, z, t)| \rangle' \\ &\lesssim \int_0^\infty \langle |\nabla'^2 \Gamma_{d-1}(\cdot, t - s)| \rangle' |\partial_z \Gamma_1(\tilde{z} + z, t - s) - \partial_z \Gamma_1(\tilde{z} - z, t - s)| \langle |(-\Delta')^{-1} \nabla' f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z} \\ (87) \&(79) \lesssim \frac{R}{(t - s)} \int_0^\infty |\partial_z \Gamma_1(\tilde{z} + z, t - s) - \partial_z \Gamma_1(\tilde{z} - z, t - s)| \langle |f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z}. \end{aligned}$$

Multiplying by the weight $\frac{1}{z}$ and integrating in $z \in (0, \infty)$ we get

$$\int_0^\infty \langle |\nabla' \partial_z u_{N_s}(\cdot, z, t)| \rangle' \frac{dz}{z} \lesssim \sup_{\tilde{z}} \int_0^\infty K_{t-s}(z, \tilde{z}) dz \begin{cases} \frac{1}{R} \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' \frac{d\tilde{z}}{\tilde{z}}, \\ \frac{1}{(t-s)} R \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' \frac{d\tilde{z}}{\tilde{z}}, \end{cases}$$

²With abuse of notation we will denote with u_{N_s} and f their even reflection

where we called $K_{t-s}(z, \tilde{z}) = \frac{\tilde{z}}{z} |\partial_z \Gamma_1(\tilde{z} - z, t - s) - \partial_z \Gamma_1(z + \tilde{z}, t - s)|$.

Recalling

$$\sup_{\tilde{z}} \int_0^\infty K_{t-s}(z, \tilde{z}) dz \stackrel{(86)}{\lesssim} \int_{\mathbb{R}} |\partial_z \Gamma_1(z, t - s)| dz + \sup_{z \in \mathbb{R}} (z^2 |\partial_z^2 \Gamma_1(z, t - s)|)$$

and observing that, in this case

$$\int_{\mathbb{R}} |\partial_z \Gamma_1(z, t - s)| dz + \sup_{z \in \mathbb{R}} (z^2 |\partial_z^2 \Gamma_1(z, t - s)|) \stackrel{(88) \& (91)}{\lesssim} \frac{1}{(t - s)^{\frac{1}{2}}},$$

we can conclude that

$$\int_0^\infty \langle |\nabla' \partial_z u_{N_s}(\cdot, t)| \rangle' \frac{dz}{z} \lesssim \left\{ \begin{array}{l} \frac{1}{(t-s)^{\frac{1}{2}}} \frac{1}{R} \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' \frac{d\tilde{z}}{\tilde{z}} \\ \frac{1}{(t-s)^{\frac{3}{2}}} R \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' \frac{d\tilde{z}}{\tilde{z}} \end{array} \right.$$

Finally, inserting (64) and integrating in time we have

$$\begin{aligned} & \int_0^\infty \int_0^\infty \langle |\nabla' \partial_z u_{N_s}(\cdot, z, t)| \rangle' \frac{dz}{z} dt \\ (64) \quad & \lesssim \int_0^\infty \int_0^\infty \int_0^t \langle |\nabla' \partial_z u_{N_s}(\cdot, \tilde{z}, t)| \rangle' \frac{dz}{z} ds dt \\ & \lesssim \int_s^\infty \int_0^\infty \min\left\{ \frac{1}{R(t-s)^{\frac{1}{2}}}, \frac{R}{(t-s)^{\frac{3}{2}}} \right\} \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' \frac{d\tilde{z}}{\tilde{z}} ds dt \\ (67) \& (68) \quad & \lesssim \int_0^\infty \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' \frac{d\tilde{z}}{\tilde{z}} ds. \end{aligned}$$

Argument for (61):

Recall that we need to prove

$$\sup_z \int_0^\infty |\nabla' \partial_z u_C| dt \lesssim \langle |\nabla' \partial_z u|_{z=0} \rangle'.$$

By equation (59), the even extension $\overline{u_C}$ satisfies

$$(\partial_t - \Delta) \overline{u_C} = -[\partial_z \overline{u_C}] \delta_{z=0} = -2\partial_z u_C \delta_{z=0} = -2\partial_z u|_{z=0} \delta_{z=0} \quad (73)$$

and therefore we study the following problem on the whole space

$$\begin{cases} (\partial_t - \Delta) \overline{u_C} &= -2\partial_z u|_{z=0} \delta & \text{for } z \in \mathbb{R}, t > 0, \\ \overline{u_C} &= 0 & \text{for } t = 0. \end{cases} \quad (74)$$

By Duhamel's principle

$$\overline{u_C}(x', z, t) = \int_{s=0}^t \overline{u_{C_s}}(x', z, t) ds, \quad (75)$$

where $\overline{u_{C_s}}$ solves the initial value problem

$$\begin{cases} (\partial_t - \Delta) \overline{u_{C_s}} = 0 & \text{for } z \in \mathbb{R}, t > s, \\ \overline{u_{C_s}} = -2\partial_z u|_{z=0} \delta & \text{for } z \in \mathbb{R}, t = s. \end{cases} \quad (76)$$

The solution of problem (76) can be represented via the heat kernel as

$$\begin{aligned} \overline{u_{C_s}}(x', z, t) &= \int \Gamma(z - \tilde{z}, t - s) *_{x'} (-2\partial_z u|_{z=0} \delta)(\tilde{z}, s) d\tilde{z}, \\ &= -2\Gamma(z, t - s) *_{x'} \partial_z u(z, s)|_{z=0}. \end{aligned}$$

We apply $\nabla' \partial_z$ to the representation above

$$\nabla' \partial_z \overline{u_{C_s}}(x', z, t) = \int_{\mathbb{R}^{d-1}} -2\Gamma_{d-1}(x' - \tilde{x}', t - s) \partial_z \Gamma_1(z, t - s) \nabla' \partial_z u(\cdot, z, s)|_{z=0} d\tilde{x}'$$

and then average in the horizontal direction,

$$\begin{aligned} &\langle |\nabla' \partial_z \overline{u_{C_s}}(x', z, t)| \rangle' \\ &\lesssim \langle |\Gamma_{d-1}(x', t - s)| \rangle' |\partial_z \Gamma_1(z, t - s)| \langle |\nabla' \partial_z u(\cdot, z, s)|_{z=0} \rangle' \\ (87) \quad &\lesssim |\partial_z \Gamma_1(z, t - s)| \langle |\nabla' \partial_z u(\tilde{x}', z, s)|_{z=0} \rangle'. \end{aligned}$$

Inserting the previous estimate in the Duhamel formula 75 and integrating in time we get

$$\begin{aligned} &\int_0^\infty \langle |\nabla' \partial_z \overline{u_C}(x', z, t)| \rangle' dt \\ &\leq \int_0^\infty \int_0^t \langle |\nabla' \partial_z \overline{u_{C_s}}(x', z, t)| \rangle' ds dt \\ &\lesssim \int_0^\infty \int_s^\infty |\partial_z \Gamma_1(z, t - s)| dt \langle |\nabla' \partial_z u(\tilde{x}', z, s)|_{z=0} \rangle' ds \\ (89) \quad &\lesssim \int_0^\infty \langle |\nabla' \partial_z u(\tilde{x}', z, s)|_{z=0} \rangle' ds. \end{aligned} \quad (77)$$

The estimate (61) follows immediately after passing to the supremum in (77). \square

4 Appendix

4.1 Preliminaries

We start this section by proving some elementary bounds and equivalences, coming directly from the definition of horizontal bandedness (93). These will turn to be crucial in the proof of the main result.

Lemma 4.

a) *If*

$$\mathcal{F}'r(k', z, t) = 0 \quad \text{unless} \quad R|k'| \geq 4 \quad (78)$$

then

$$\langle |r(\cdot, z, t)| \rangle' \leq R \langle |\nabla' r(\cdot, z, t)| \rangle'. \quad (79)$$

In particular

$$\|r\|_{(0,\infty)} \leq R \|\nabla' r\|_{(0,\infty)}.$$

b) *If*

$$\mathcal{F}'r(k', z, t) = 0 \quad \text{unless} \quad R|k'| \leq 1 \quad (80)$$

then

$$\langle |\nabla' r(\cdot, z, t)| \rangle' \leq \frac{1}{R} \langle |r(\cdot, z, t)| \rangle'. \quad (81)$$

In particular

$$\|\nabla' r\|_{(0,\infty)} \leq \frac{1}{R} \|r\|_{(0,\infty)}.$$

c) *If*

$$\mathcal{F}'r(k', z, t) = 0 \quad \text{unless} \quad 1 \leq R|k'| \leq 4$$

then

$$\|\nabla'(-\Delta')^{-\frac{1}{2}}r\|_{(0,\infty)} \sim \|r\|_{(0,\infty)}, \quad (82)$$

and

$$\|(-\Delta')^{\frac{1}{2}}r\|_{(0,\infty)} \sim \|\nabla' r\|_{(0,\infty)}. \quad (83)$$

Remark 3. *All the results stated in Lemma 4 are valid with the norm $\|\cdot\|_{(0,\infty)}$ replaced with $\|\cdot\|_{(0,1)}$.*

Remark 4. Notice that from (82) and (83), it follows

$$\|\nabla'(-\Delta')^{-1}\nabla' \cdot r\|_{(0,\infty)} \lesssim \|r\|_{(0,\infty)}. \quad (84)$$

Proof.

a) By rescaling we may assume $R = 1$.

Let $\phi \in \mathcal{S}(\mathbb{R}^{d-1})$ be a Schwartz function such that

$$\mathcal{F}'\phi(k') = \begin{cases} 0 & \text{for } |k'| \geq 1 \\ 1 & \text{for } |k'| \leq 1 \end{cases}$$

and such that $\int_{\mathbb{R}^{d-1}} \phi(x') dx' = 1$.

We claim that, under assumption (78), there exists $\psi \in L^1(\mathbb{R}^{d-1})$ such that

$$(\text{Id} - \phi *')r = \psi *' \nabla r. \quad (85)$$

Since $r = r - \phi * r$, if we assume (85) the conclusion follows from Young's inequality

$$\int_{\mathbb{R}^{d-1}} |r(x', z)| dx' \leq \int_{\mathbb{R}^{d-1}} |\psi(x')| dx' \int_{\mathbb{R}^{d-1}} |\nabla r(x', z)| dx'.$$

Argument for (85):

Using the assumptions on ϕ and performing suitable change of vari-

ables, we find

$$\begin{aligned}
& r(x', z) - \int \phi(x' - y') r(y', z) dy' \\
&= \int \phi(x' - y') (r(x', z) - r(y', z)) dy' \\
&= \int_{\mathbb{R}^{d-1}} \phi(x' - y') \int_0^1 (x' - y') \nabla' r(tx' + (t-1)(x' - y'), z) dy' dt \\
&= \int_0^1 \int_{\mathbb{R}^{d-1}} \phi(\xi) \nabla' r(x' + (t-1)\xi, z) \cdot \xi d\xi dt \\
&= \int_0^1 \int_{\mathbb{R}^{d-1}} \phi\left(\frac{\hat{y}' - x'}{t}\right) \nabla r(\hat{y}', z) \cdot \frac{\hat{y}' - x'}{t} dt \frac{1}{t^{d-1}} d\hat{y}' \\
&= \int_{\mathbb{R}^{d-1}} \nabla' r(\hat{y}', z) \cdot \left(\int_0^1 \phi\left(\frac{\hat{y}' - x'}{t}\right) \frac{\hat{y}' - x'}{t^d} dt \right) d\hat{y}' \\
&= \int_{\mathbb{R}^{d-1}} \nabla' r(\hat{y}', z) \psi\left(\frac{\hat{y}' - x'}{t}\right) d\hat{y}',
\end{aligned}$$

where

$$\psi(x') = \int_0^1 \phi\left(\frac{-x'}{t}\right) \frac{x'}{t^d} dt.$$

We notice that $\psi \in L^1(\mathbb{R}^{d-1})$, in fact

$$\int_{\mathbb{R}^{d-1}} |\psi(x')| dx' \leq \int_0^1 \int_{\mathbb{R}^{d-1}} \left| \phi(x'/t) \frac{x'}{t^d} \right| dx' dt = \int_{\mathbb{R}^{d-1}} |\phi(\xi) \xi| d\xi.$$

b) In Fourier space we have

$$\mathcal{F}' \nabla' r(k', z) = ik' \mathcal{F}' r(k', z) = R^{-1} \mathcal{F}' G(Rk') \mathcal{F}' r(k', z) = R^{-1} \mathcal{F}' G_R(k') \mathcal{F}' r(k', z),$$

where G is a Schwartz function and $G_R(x') = R^{-d} \mathcal{F}' G(x'/R)$. Since $\int |G_R| dx' = \int |G| dx'$ is independent of R , we may conclude by Young

$$\int |\nabla' r| dx' \leq \frac{1}{R} \int |G_R| dx' \int |r| dx' \lesssim \frac{1}{R} \int |r| dx'.$$

□

Here we prove an elementary estimate that will be applied in the argument for (53) and (60), Lemma 3

Lemma 5.

Let $K = K(z)$ be a real function and define

$$\overline{K}(z, \tilde{z}) = \frac{\tilde{z}}{z} |K(\tilde{z} - z) - K(z + \tilde{z})|.$$

Then

$$\sup_{\tilde{z}} \int_0^\infty \overline{K}(z, \tilde{z}) dz \lesssim \int_{\mathbb{R}} |K(z)| dz + \sup_{z \in \mathbb{R}} (z^2 |\partial_z K(z)|). \quad (86)$$

Proof. Let us distinguish two regions: $\frac{1}{2} \left| \frac{\tilde{z}}{z} \right| < 1$ and $\frac{1}{2} \left| \frac{\tilde{z}}{z} \right| > 1$.
For $|z| \geq \frac{1}{2} |\tilde{z}|$ we have

$$\begin{aligned} & \sup_{\tilde{z}} \int_{|z| \geq \frac{1}{2} |\tilde{z}|} |\overline{K}(z, \tilde{z})| dz \\ & \leq \max_{\tilde{z}} \int_{|z| \geq \frac{1}{2} |\tilde{z}|} |K(\tilde{z} - z) - K(z + \tilde{z})| dz \lesssim \int |K(z)| dz. \end{aligned}$$

While for the region $|z| \leq \frac{1}{2} |\tilde{z}|$ we have,

$$\begin{aligned} & \max_{\tilde{z}} |\tilde{z}| \int_{|z| \leq \frac{1}{2} |\tilde{z}|} \frac{1}{|z|} |K(\tilde{z} - z) - K(z + \tilde{z})| dz \\ & = \max_{\tilde{z}} |\tilde{z}| \int_{|z| \leq \frac{1}{2} |\tilde{z}|} \frac{1}{|z|} \left| \int_{-1}^1 K'(\tilde{z} + tz) z dt \right| dz \\ & \leq \max_{\tilde{z}} |\tilde{z}| \int_{-1}^1 \frac{1}{t} \int_{|z| \leq \frac{t}{2} |\tilde{z}|} |K'(\tilde{z} + z)| dz dt \\ & \stackrel{\frac{1}{2} |\tilde{z}| \leq |\tilde{z} + z|}{\leq} \max_{\tilde{z}} \int_{-1}^1 \frac{1}{t} \int_{|z| \leq \frac{t}{2} |\tilde{z}|} 2 |\tilde{z} + z| |K'(\tilde{z} + z)| dt dz \\ & \leq \max_{\tilde{z}} \int_{-1}^1 \frac{2}{t} \max_{|z| \leq \frac{t}{2} |\tilde{z}|} \{ |\tilde{z} + z| |K'(\tilde{z} + z)| \} \left(\int_{|z| \leq \frac{t}{2} |\tilde{z}|} dz \right) dt \\ & = \max_{\tilde{z}} \int_{-1}^1 \frac{1}{t} \max_{|z| \leq \frac{t}{2} |\tilde{z}|} \{ |\tilde{z} + z| |K'(\tilde{z} + z)| \} t |\tilde{z}| dt \\ & = 2 \max_{\tilde{z}} |\tilde{z}| \max_{|z| \leq \frac{t}{2} |\tilde{z}|} \{ |\tilde{z} + z| |K'(\tilde{z} + z)| \} \\ & \stackrel{\frac{1}{2} |\tilde{z}| \leq |\tilde{z} + z|}{\leq} 4 \max_{\tilde{z}} \max_{|z| \leq \frac{t}{2} |\tilde{z}|} \{ |z + \tilde{z}|^2 |K'(\tilde{z} + z)| \}. \end{aligned}$$

In conclusion we have

$$\max_z \int |\bar{K}(z, \tilde{z})| dz \lesssim \int |K(z)| dz + \max_z |z|^2 |K'(z)|.$$

□

4.2 Heat kernel: elementary estimates

In this section we recall the definition of the heat kernel and some properties and estimates that we will use throughout the paper.

The function $\Gamma : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ is defined as

$$\Gamma(x, t) = \frac{1}{t^{d/2}} \exp\left(-\frac{|x|^2}{4t}\right)$$

and we can rewrite it as

$$\Gamma(x, t) = \Gamma_1(z, t) \Gamma_{d-1}(x', t) \quad x' \in \mathbb{R}^{d-1}, z \in \mathbb{R},$$

where

$$\Gamma_1(z, t) = \frac{1}{t^{1/2}} \exp\left(-\frac{z^2}{4t}\right)$$

and

$$\Gamma_{d-1}(z, t) = \frac{1}{t^{(d-1)/2}} \exp\left(-\frac{|x'|^2}{4t}\right).$$

Here we list the bounds on the derivatives of Γ that are used in Section 4, Lemma 3:

1.

$$\langle |(\nabla')^n \Gamma_{d-1}| \rangle' \approx \frac{1}{t^{\frac{n}{2}}}. \quad (87)$$

2.

$$\int_{\mathbb{R}} |\partial_z^n \Gamma_1| dz \lesssim \frac{1}{t^{\frac{n}{2}}}. \quad (88)$$

3.

$$\int_0^\infty |\partial_z \Gamma_1(z, t)| dt = \int_0^\infty \left| \frac{1}{\hat{t}^{3/2}} \exp\left(-\frac{1}{4\hat{t}}\right) \right| d\tilde{t} \lesssim 1, \quad (89)$$

where we have used the change of variable $\hat{t} = \frac{t}{z^2}$.

4.

$$\sup_{z \in \mathbb{R}} (z |\partial_z \Gamma_1(z, t)|) = \sup_{\xi} \left| \frac{1}{t^{\frac{1}{2}}} \xi^2 \exp^{-\xi^2} \right| \lesssim \frac{1}{t^{\frac{1}{2}}}, \quad (90)$$

where we have used the change of variable $\xi = \frac{z}{t^{\frac{1}{2}}}$.

5.

$$\sup_{z \in \mathbb{R}} (z^2 |\partial_z \Gamma_1(z, t)|) = \sup_{\xi} \left| \xi^3 \exp^{-\xi^2} \right| \lesssim 1, \quad (91)$$

where we have used the change of variable $\xi = \frac{z}{t^{\frac{1}{2}}}$.

5 Notations

The $(d-1)$ -dimensional torus:

We denote with $[0, L)^{d-1}$ the $(d-1)$ -dimensional torus of lateral size L .

The spatial vector:

$$x = (x', z) \in [0, L)^{d-1} \times \mathbb{R}.$$

The horizontal average:

$$\langle \cdot \rangle' = \frac{1}{L^{d-1}} \int_{[0, L)^{d-1}} \cdot \, dx'.$$

Long-time and horizontal average:

$$\langle \cdot \rangle = \limsup_{t_0 \rightarrow \infty} \frac{1}{t_0} \int_0^{t_0} \langle \cdot \rangle' dt. \quad (92)$$

Convolution in the horizontal direction:

$$f *_{x'} g(x') = \int_{[0, L)^{d-1}} f(x' - \tilde{x}') g(\tilde{x}') d\tilde{x}'.$$

Convolution in the whole space:

$$f * g(x) = \int_{\mathbb{R}} \int_{[0, L)^{d-1}} f(x' - \tilde{x}', z - \tilde{z}) g(\tilde{x}', \tilde{z}) d\tilde{x}' d\tilde{z}.$$

Horizontal Fourier transform:

$$\mathcal{F}'f(k', z, t) = \frac{1}{L^{d-1}} \int e^{-ik' \cdot x'} f(x', z, t) dx'.$$

where k' is the conjugate variable of x' .

Horizontally band-limited function:

A function $g = g(x', z, t)$ is called *horizontally band-limited* with bandwidth R if it satisfies the *bandedness assumption*

$$\mathcal{F}'g(k', z, t) = 0 \text{ unless } 1 \leq R|k'| \leq 4 \text{ where } R < R_0. \quad (93)$$

Interpolation norms:

$$\begin{aligned} \|f\|_{(0,1)} &= \|f\|_{R;(0,1)} = \inf_{f=f_1+f_2} \left\{ \left\langle \sup_{z \in (0,1)} |f_1| \right\rangle + \left\langle \int_{(0,1)} |f_2| \frac{dz}{z(1-z)} \right\rangle \right\}, \\ \|f\|_{(0,\infty)} &= \|f\|_{R;(0,\infty)} = \inf_{f=f_1+f_2} \left\{ \left\langle \sup_{z \in (0,\infty)} |f_1| \right\rangle + \left\langle \int_{(0,\infty)} |f_2| \frac{dz}{z} \right\rangle \right\}, \\ \|f\|_{(-\infty,1)} &= \|f\|_{R;(-\infty,1)} = \inf_{f=f_1+f_2} \left\{ \left\langle \sup_{z \in (-\infty,1)} |f_1| \right\rangle + \left\langle \int_{(-\infty,1)} |f_2| \frac{dz}{1-z} \right\rangle \right\}. \end{aligned}$$

where f_0, f_1 satisfy the bandedness assumption (93).

Throughout the paper we will denote with \lesssim the inequality up to universal constants.

Acknowledgement

C.N. was supported by IMPRS of MPI MIS (Leipzig). A.C. was partially supported by Whittaker Research Fellowship.

References

- [1] Antoine Choffrut, Camilla Nobili, and Felix Otto. Upper bounds on Nusselt number at finite Prandtl number.
- [2] Vsevolod Alekseevich Solonnikov. Estimates for solutions of a non-stationary linearized system of Navier-Stokes equations. *Trudy Matematicheskogo Instituta im. VA Steklova*, 70:213–317, 1964.

- [3] Seiji Ukai. A solution formula for the Stokes equation in \mathbb{R}_+^n . *Communications on Pure and Applied Mathematics*, 40(5):611–621, 1987.
- [4] Paolo Maremonti and Vsevolod A Solonnikov . On nonstationary Stokes problem in exterior domains. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, 24(3):395–449, 1997.
- [5] Giovanni P Galdi. *An introduction to the mathematical theory of the Navier-Stokes Equations: steady-state problems*, volume 501. Springer, 2011.